1. (20) Define the following:

a. $\lim_{(x,y)\to(1,2)} f(x,y) = 3,$

Means that for any $\epsilon > 0$, there is a $\delta > 0$ such that if $0 < \|(x,y) - (1,2)\| < \delta$, then $|f(x,y) - 3| < \epsilon$

b. f is differentiable at the point $(1,2)$,

Means there are numbers ϵ_1 and ϵ_2 such that

$$f(x+\Delta x, y+\Delta y) = f(x,y) + \left.\frac{\partial f}{\partial x}\right|_{(1,2)} \Delta x + \left.\frac{\partial f}{\partial y}\right|_{(1,2)} \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

where $\lim_{(\Delta x,\Delta y)\to(0,0)} \epsilon_i = 0$ for $i = 1$ and $i = 2$.

c. f is continuous at the point $(1,2)$,

Means that $\lim_{(x,y)\to(1,2)} f(x,y) = f(1,2)$.

d. The partial derivative of f with respect to x at the point $(1,2)$.

$$\left.\frac{\partial f}{\partial x}\right|_{(1,2)} = \lim_{h\to 0} \frac{f(1+h,2) - f(1,2)}{h} .$$

2. (20) Let $\vec{A} = (3, 2, -1)$, $\vec{B} = (5, 7, 2)$.

a. Find the angle between $\vec{A}$ and $\vec{B}$.

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{\|\vec{A}\|\,\|\vec{B}\|} = \frac{15+14-2}{\sqrt{14}\sqrt{78}} = \frac{27}{\sqrt{14}\sqrt{78}}$$

Thus,

$$\theta = \cos^{-1}\left(\frac{27}{\sqrt{14}\sqrt{78}}\right) \approx 0.61451$$

b. Find the vector projection of $\vec{B}$ onto $\vec{A}$.

$$\text{Proj}_{\vec{A}}\vec{B} = \frac{\vec{A}\cdot\vec{B}}{\|\vec{A}\|^2} A = \frac{27}{14}(3,2,-1)$$

c. Find the area of the parallelogram determined by $\vec{A}$ and $\vec{B}$.

The area equals the magnitude of the cross product of the two vectors,

$$\text{Area} = \|\vec{A}\times\vec{B}\| = \|(11,-11,11)\|$$
$$= 11\|(1,-1,1)\| = 11\sqrt{3} .$$

d. Find an equation for the straight line passing through the points $(3, 2, -1)$, and $(5, 7, 2)$.

$$\Gamma(t) = (3,2,-1) + t[(5,7,2) - (3,2,-1)]$$
$$= (3,2,-1) + t(2,5,3)$$

3. (15) The polar coordinates r, and θ, are related to the Cartesian coordinates by the equations:

$$x = r\cos\theta$$

$$y = r\sin\theta\ .$$

a. We saw in class that $\frac{\partial r}{\partial x} = \cos\theta$, $\frac{\partial r}{\partial y} = \sin\theta$, and $\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$. Find a formula for $\frac{\partial \theta}{\partial x}$ in terms of r and θ.

Differentiating $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ with respect to x, we have

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + (y/x)^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2 + y^2}$$

$$= \frac{-r\sin\theta}{r^2} = \frac{-\sin\theta}{r}\ .$$

b. Let $f(r, \theta) = r^2 + e^{\theta}$. Find a formula for $\frac{\partial f}{\partial y}$ in terms of r and θ.

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$= 2r\sin\theta + e^{\theta}\frac{\cos\theta}{r}\ .$$

4. (25) Let $f(x,y,z) = x^2 - 3y^2 + yz - 2z$.

a. Use the definition of partial derivative to compute $\frac{\partial f}{\partial z}$ at the point $(-1, 2, 1)$.

$$\left.\frac{\partial f}{\partial z}\right|_{(-1,2,1)} = \lim_{h \to 0} \frac{f(-1,2,1+h) - f(-1,2,1)}{h}$$

$$= \lim_{h \to 0} \frac{[1 - 12 + 2(1+h) - 2(1+h)] - (-11)}{h}$$

$$= \lim_{h \to 0} \frac{-11 - (-11)}{h} = 0 .$$

b. Find the directional derivative of f in the direction $(3, 5, \sqrt{2})$ at the point $(-1, 2, 1)$.

$$D_N(f)|_{(-1,2,1)} = \nabla f|_{(-1,2,1)} \cdot \frac{(3, 5, \sqrt{2})}{6}$$

$$= \left.\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\right|_{(-1,2,1)} \cdot \frac{(3, 5, \sqrt{2})}{6}$$

$$= (2x, -6y + z, y - 2)|_{(-1,2,1)} \cdot \frac{(3, 5, \sqrt{2})}{6}$$

$$= (-2, -11, 0) \cdot \frac{(3, 5, \sqrt{2})}{6}$$

$$= \frac{-6 - 55}{6} = -\frac{61}{6} .$$

c. At the point $(-1, 2, 1)$ in what direction will f have the greatest rate of change?

The greatest rate of change of f is in the direction of the gradient of f. Thus, at the point $(-1, 2, 1)$ the gradient is $(-2, -11, 0)$, and this is the direction of maximum rate of change of f.

d. Find an equation for the plane tangent to the surface $f = -11$ at the point $(-1, 2, 1)$.

Since the gradient is perpendicular to level surfaces we know that $(-2, -11, 0)$ is normal to this surface at the point $(-1, 2, 1)$. Hence an equation for the tangent plane to the surface at this point is

$$(x + 1, y - 2, z - 1) \cdot (-2, -11, 0) = 0$$

$$2x + 11y = 20 .$$

e. Let $\Gamma(t) = (x(t), y(t), z(t))$ be a curve which passes through the point $(-1, 2, 1)$ when $t = 3/4$. Suppose $\Gamma'(3/4) = (3, 5, \sqrt{2})$. Set $g(t) = f(\Gamma(t))$. What must $g'(3/4)$ equal?

$$g'(t) = \nabla f|_{\Gamma(3/4)} \cdot \Gamma'(3/4)$$

$$= \nabla f|_{(-1,2,1)} \cdot (3, 5, \sqrt{2})$$

$$= (-2, -11, 0) \cdot (3, 5, \sqrt{2}) = -6 - 55 = -61$$

5. (10) Let $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$.

a. What is the domain of f?

The domain of f is the set $\{(x,y) : x^2+y^2 \neq 0\} = R^2 \backslash \{(0,0)\}$.

b. Find $\lim_{(x,y)\to(3,4)} f(x,y)$

$$\lim_{(x,y)\to(3,4)} f(x,y) = \frac{12}{\sqrt{9+16}} = \frac{12}{5}.$$

c. Find $\lim_{(x,y)\to(0,0)} f(x,y)$

This is a bit harder as the limit point $(0,0)$ is not in the domain of the function, which means continuity is not applicable. However, if we switch to polar coordinates we have

$$\begin{aligned}\lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} \\ &= \lim_{r\to 0} \frac{(r\cos\theta)(r\sin\theta)}{r} \\ &= \lim_{r\to 0} r\cos\theta\sin\theta = 0.\end{aligned}$$

6. (10) Let $f(x,y) = x^3 - 3xy + y^3$

a. Find all singular points. That is, find all possible locations of local extrema.

The gradient of f equals $\nabla f = \left(3x^2 - 3y, -3x + 3y^2\right)$. Setting both components equal to zero we have the following system of equations

$$\begin{aligned} x^2 - y &= 0 \\ -x + y^2 &= 0.\end{aligned}$$

This leads to the equation $-x + x^4$, which has the two real solutions $x = 0$ and $x = 1$. Sincy $y = x^2$ we have as our singular points

$$(0,0) \text{ and } (1,1).$$

b. At each of the points you found in part b. determine what sort of singular point it is.

$$\frac{\partial^2 f}{\partial x^2} = 6x,\ \frac{\partial^2 f}{\partial y^2} = 6y,\ \frac{\partial^2 f}{\partial x \partial y} = -3.$$

Thus, the expression $D = \dfrac{\partial^2 f}{\partial x^2}\dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial^2 f}{\partial x \partial y}\right)^2 = 36xy - 9$. Evaluating D at the critical points we have

$D(0,0) = -9$ implies that the origin is a saddle point.

Since $\dfrac{\partial^2 f}{\partial x^2}$ is positive at $(1,1,)$ and $D(1,1) = 27$ we know that $(1,1)$ is a local minimum, .