Math 221 Solutions Exam 2 November 1, 2006

1. (15) Define the following

a. $f(x,y)$ is differentiable at the point $(1,-2)$.

This means there are numbers ϵ_i such that

$$f(1+\Delta x,-2+\Delta y) = f(1,-2) + \left.\frac{\partial f}{\partial x}\right|_{(1,-2)} \Delta x + \left.\frac{\partial f}{\partial y}\right|_{(1,-2)} \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

and $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \epsilon_i = 0$ for $i = 1, 2$.

b. $f(x,y)$ has a local maximum at the point $(1,-2)$.

This means that there is a disk, D, surrounding the point $(1,-2)$ such for any $(x,y) \in D$ we have

$$f(x,y) \leq f(1,-2).$$

c. What do the spherical variables ρ, θ, and ϕ represent ?

Given a point $P = (x,y,z) \in R^3$, let L represent the line segment from the origin to the point P. Then the spherical variables represent

$\rho = (x^2+y^2+z^2)^{1/2}$ the distance from P to the origin

$\phi =$ the angle determined by the positive z-axis and the line L

$\theta =$ the angle between the positive x-axis and the lined formed by projecting L onto the x-y plane.

2. (15) The coordinates of a point in one coordinate system are given, find the coordinates of that point in the coordinate system specified.

a. $(1,-2)$ are the polar coordinates (r,θ) of a point, find its Cartesian coordinates.

$$x = r\cos\theta = 1\cos(-2) = \cos(-2)$$
$$y = r\sin\theta = \sin(-2)\ .$$

b. $(1,1,2)$ are the spherical coordinates (ρ,θ,ϕ) of a point, find its cylindrical coordinates.

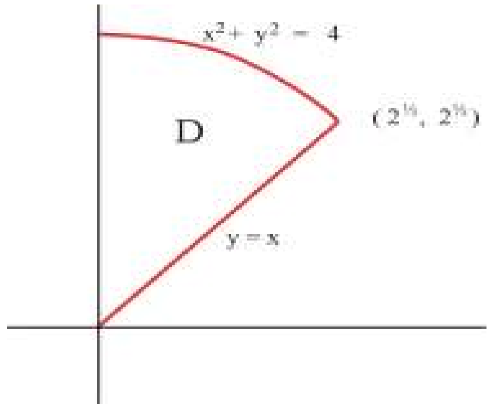
$$r = \rho\sin\phi = \sin 2$$
$$\theta = 1$$
$$z = \rho\cos\phi = \cos 2$$

c. $(1,1,2)$ are the Cartesian coordinates of a point in R^3. What are its spherical coordinates ?

$$\rho = (1+1+2^2)^{1/2} = \sqrt{6}$$
$$\phi = \cos^{-1}\left(\frac{2}{\sqrt{6}}\right)$$
$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}1 = \frac{\pi}{4}.$$

3. (30) The iterated integral $\int\int_D dA = \int_0^{\sqrt{2}} dx \int_x^{\sqrt{4-x^2}} dy$ has a value equal to the area of a region D in R^2.

a. Sketch the region D.



b. Express the area of the region D by interchanging the order of integration in the given iterated integral.

$$\int\int_D dA = \int_0^{\sqrt{2}} dy \int_0^y dx + \int_{\sqrt{2}}^2 dy \int_0^{\sqrt{4-y^2}} dx \, .$$

c. Express $\int\int_D dA$ in terms of an iterated integral in polar coordinates.

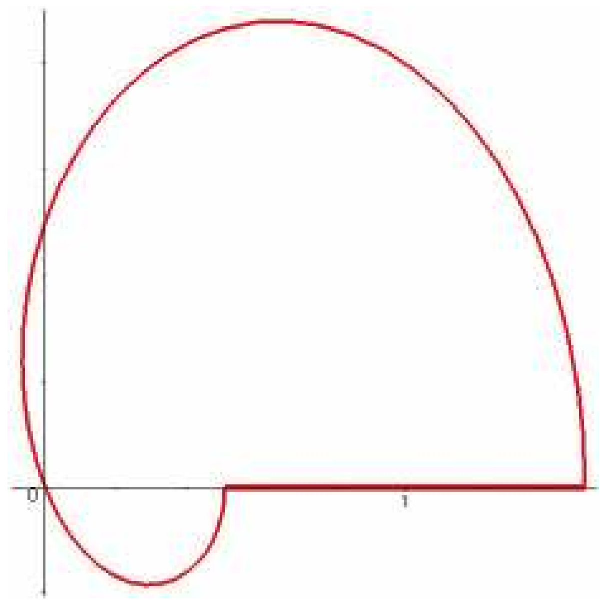
$$\int_{\pi/4}^{\pi/2} d\theta \int_0^2 r dr \, .$$

d. What is the area of D ?

$$\text{Area} = \frac{1}{8}\left(\pi(2)^2\right) = \frac{\pi}{2} \, .$$

4. (20) The curve $r = \frac{1}{2} + \cos\theta$, for $0 \le \theta \le \pi$, and that part of the x-axis for $x \ge 1/2$ encloses a region D in R^2.

a. Sketch the region D.



b. Set up an integral whose value equals the area of D. Note: your answer must be in terms of iterated integrals.

$$\text{Area} = \int_0^{2\pi/3} d\theta \int_0^{1/2+\cos\theta} r\,dr + \int_{2\pi/3}^{\pi} d\theta \int_0^{|1/2+\cos\theta|} r\,dr\,.$$

c. What is the slope of the tangent line to this curve when $\theta = \frac{\pi}{2}$?

$$\text{slope} = \frac{dy}{dx} = \frac{dy/d\theta}{dy/d\theta}$$

$$= \left.\frac{-\sin\theta\sin\theta + (1/2 + \cos\theta)\cos\theta}{-\sin\theta\cos\theta - (1/2 + \cos\theta)\sin\theta}\right|_{\theta=\pi/2}$$

$$= \frac{-1}{-1/2} = 2\,.$$

5. (20) Let E denote the region in R^3 that is bounded by the sphere $x^2+y^2+z^2 = 4$, and the cylinder $(x-1)^2+y^2 = 1$

a. Set up the iterated integrals needed to determine the volume of E Your choice of coordinates.

$$\text{Volume} = \int_0^2 dx \int_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}} dy \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} dz$$

b. Set up the iterated integrals needed to calculate the surface area of that part of this region that actually lies on the sphere $x^2+y^2+z^2 = 4$. Your choice of coordinate system.

$$\begin{aligned}\text{Surface area} &= 2\int_0^2 dx \int_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dy \\ &= 2\int_0^2 dx \int_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}} \sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}}\, dy \\ &= 4\int_0^2 dx \int_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}} (4-x^2-y^2)^{-1/2}\, dy\end{aligned}$$