1. (10)

a. Define $\lim_{(x,y)\to(-2,3)} f(x,y) = 7$,

This means that for any $\epsilon > 0$, there is a $\delta > 0$ such that if

$$0 < \sqrt{(x+2)^2 + (y-3)^2} < \delta, \text{ then } |f(x,y) - 7| < \epsilon.$$

b. state the Divergence Theorem.

Let E be a bounded region in R^3 whose boundary S consists of a finite number of smooth orientable surfaces. Let $\vec{F}$ be a twice continuously differentiable vector field defined on a region containg E, the

$$\iiint_E \operatorname{div}\left(\vec{F}\right)\, dV = \iint_S \vec{F}\cdot d\vec{S}.$$

2. (10) Let $f(x,y) = x^2 - 5xy$.

a. Define the directional derivative of f at the point $(-5,2)$ in the direction $(1,3)$.

$$D_N f = \lim_{h\to 0} \frac{f\left(-5 + h/\sqrt{10}, 2 + 3h/\sqrt{10}\right) - f\left(-5,2\right)}{h}.$$

b. Compute the directional derivative of f at the point $(-5,2)$ in the direction $(1,3)$. You do not have to use the definition of the directional derivative to answer this.

$$\begin{aligned} D_N f &= \operatorname{grad}(f)\cdot\left(1/\sqrt{10}, 3/\sqrt{10}\right) \\ &= \left(2x - 5y, \ -5x\right)\Big|_{(-5,2)}\cdot\left(1/\sqrt{10}, 3/\sqrt{10}\right) \\ &= (-20, 25)\cdot\left(1/\sqrt{10}, 3/\sqrt{10}\right) \\ &= \frac{-20+75}{\sqrt{10}} = \frac{55}{\sqrt{10}}. \end{aligned}$$

3. (20) Let $\vec{F}\left(x,y,z\right) = \left(x^2 y, 2x - y + z, x^2\right)$. Let C be the path consisting of straight lines joining the points, in the given order, $\left(0,0,0\right) \to \left(1,0,0\right) \to \left(1,1,0\right) \to \left(1,1,1\right)$. Compute $\int_C \vec{F}\cdot d\vec{\Gamma}$.

Let C_1, C_2, and C_3 denote the three straight line segments that make up the path C. Parametrizations of these paths are respectively $(t,0,0)$, $(1,t,0)$, and $(1,1,t)$, where $0 \le t \le 1$ in each of these three. Then we have

$$\begin{aligned} \int_C \vec{F}\cdot d\vec{\Gamma} &= \int_{C_1} \vec{F}\cdot d\vec{\Gamma} + \int_{C_2} \vec{F}\cdot d\vec{\Gamma} + \int_{C_3} \vec{F}\cdot d\vec{\Gamma} \\ &= \int_0^1 (0, 2t, t^2)\cdot(1,0,0)dt + \int_0^1 (t, 2-t, 1)\cdot(0,1,0)dt \\ &\quad + \int_0^1 (1, 1+t, 1)\cdot(0,0,1)dt \\ &= 0 + \int_0^1 (2-t)dt + \int_0^1 dt = \frac{3}{2} + 1 \\ &= \frac{5}{2}. \end{aligned}$$

4. (20) Let $\vec{F}(x, y, z) = (\sin y, x\cos y - z\sin yz, -y\sin yz)$.

a. $\operatorname{div}(\vec{F}) =$

$$\operatorname{div}(\vec{F}) = -x\sin y - z^2\cos yz - y^2\cos yz$$

b. $\operatorname{curl}(\vec{F}) =$

$$\begin{aligned}\operatorname{curl}(\vec{F}) &= \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ \sin y & x\cos y - z\sin yz & -y\sin yz \end{bmatrix} \\ &= ((-\sin yz - yz\cos yz) - (-\sin yz - yz\sin yz), 0 - 0, \cos y - \cos y) \\ &= \vec{0}\end{aligned}$$

c. Is $\vec{F}$ a conservative force field. If yes, find a potential function Φ, such that $\nabla\Phi = \vec{F}$, if no explain why $\vec{F}$ is not conserative.

Since $\operatorname{curl}(\vec{F}) = \vec{0}$ and the domain of $\vec{F}$ is all of R^3, $\vec{F}$ is conservative. We are looking for a function Φ such that

$$\frac{\partial\Phi}{\partial x} = \sin y,\ \frac{\partial\Phi}{\partial y} = x\cos y - z\sin yz, \text{ and } \frac{\partial\Phi}{\partial z} = -y\sin yz\ .$$

The solution to the first equation is $\Phi(x,y,z) = x\sin y + f(y,z)$. Putting this into the second equation we see that $\frac{\partial f}{\partial y} = -z\sin yz$. Hence $f(x,y) = \cos yz + h(z)$, or $\Phi(x,y,z) = x\sin y + \cos yz + h(z)$. Inserting this expression for Φ into the third equation we see that $h'(z) = 0$, and $h(z) = c$ a constant. Thus,

$$\Phi(x,y,z) = x\sin y + \cos yz + c\ .$$

5. (20) Let S be that part of the surface $x + y^2 + 2z^2 = 4$ that lies in front of the $x = 0$, i.e., $x > 0$. Let $\rho(x, y, z) = xy^2 + z^2$ be the mass density function.

a. Express the mass of S as an iterated integral. You do not need to evaluate the integral.

The projection of S onto the y-z plane is the set of points (y, z) such that $y^2 + 2z^2 \leq 4$. Call this set D, then we can think of S as the graph of the function $x = 4 - y^2 - 2z^2$ where $(x, y) \in D$. A parametric representation of S is

$$r(y, z) = (4 - y^2 - 2z^2, y, z) .$$

Taking partial derivatives and the cross product we have

$$\frac{\partial r}{\partial y} = (-2y, 1, 0) \text{ and } \frac{\partial r}{\partial z} = (-4z, 0, 1)$$

$$\frac{\partial r}{\partial y} \times \frac{\partial r}{\partial z} = (1, 2y, 4z). \text{ Thus, } dS = \sqrt{1 + 4y^2 + 16z^2}\ dy\,dz .$$

Finally the mass of this surface is

$$\begin{aligned} \text{mass} &= \int\int_S (xy^2 + z^2) dS \\ &= \int\int_D [(4 - y^2 - 2z^2)y^2 + z^2] \sqrt{1 + 4y^2 + 16z^2}\ dy\,dz \\ &= \int_{-2}^{2} dy \int_{-\sqrt{(4-y^2)/2}}^{\sqrt{(4-y^2)/2}} \int_D [(4 - y^2 - 2z^2)y^2 + z^2] \sqrt{1 + 4y^2 + 16z^2}\ dz \end{aligned}$$

b. Find an equation for the tangent plane to this surface at the point $(7/4,\ 1/2,\ 1)$.

A normal to the surface is given by $\frac{\partial r}{\partial y} \times \frac{\partial r}{\partial z} = (1, 2y, 4z)$. Evaluating this at $y = 1/2$, $z = 1$ give $\vec{n} = (1, 1, 4)$. An equation for the tangent plane is then

$$[(x, y, z) - (7/4,\ 1/2,\ 1)] \cdot (1, 1, 4) = 0$$

$$x + y + 4z = \frac{25}{4} .$$

6. (20) Let $\vec{F} = \left(ye^{z^2}, y^2, e^{xy}\right)$, and let S be the surface of the solid bounded by the cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z = y - 3$. Find the outward flux of $\vec{F}$ across the surface S.

The easiest way to do this is to use the divergence theorem, which says that

$$\text{flux}\left(\vec{F}\right) = \iint_{\partial E} \vec{F} \cdot d\vec{S} = \iiint_E \text{div}\vec{F}\, dV$$

$$= \int_{-3}^{3} dx \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy \int_{y-3}^{0} (2y)\, dz$$

$$= \int_{-3}^{3} dx \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 2y(3 - y)dy$$

$$= -\int_{-3}^{3} \frac{4}{3}(9 - x^2)^{\frac{3}{2}}\, dx$$

$$= -\frac{81}{2}\pi\ .$$

Note, the limits on the z-integral. $y - 3$ is ≤ 0 for all y in the disk $x^2 + y^2 \leq 9$.