

1. (20) Define the following:

a.  $\lim_{(x,y,z) \rightarrow (-2,1,3)} f(x,y,z) = 7$

Means that for any  $\epsilon > 0$  there is a  $\delta > 0$  such that if  $\|(x+2, y-1, z-3)\| < \delta$  then  $|f(x,y,z) - 7| < \epsilon$ .

b. the directional derivative of  $f$  at the point  $(2, 3)$  in the direction  $(-1, 5)$

$$D_N(f) = \lim_{h \rightarrow 0} \frac{f(2 - h/\sqrt{26}, 3 + 5h/\sqrt{26}) - f(2, 3)}{h}.$$

c.  $f$  has a local minimum at the point  $(1, -2, 3)$

There is a  $\delta > 0$  such that if the point  $(x, y, z)$  is within  $\delta$  of the point  $(1, -2, 3)$ , then  $f(x, y, z) \geq f(1, -2, 3)$ .

d. the flux of a force field  $F$  across a surface  $S$ .

The flux is the integral of the normal component of  $F$  over the surface  $S$ , where normal means perpendicular to the surface.

2. (40) Let  $S$  be a surface with  $r(u, v) = (u + v, 2u - v^2, v + \ln u)$  for  $1 \leq u \leq 2$  and  $0 \leq v \leq 2$ .

a. compute  $d\vec{S}$  with orientation in the positive  $z$  direction,

$$\begin{aligned} d\vec{S} &= \pm \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} du dv \\ &= \pm(1, 2, 1/u) \times (1, -2v, 1) du dv \\ &= \pm(2 + 2v/u, 1/u - 1, -2v - 2) du dv. \end{aligned}$$

For  $u = 1$  and  $v = 1$  the vector about has  $z$  component equal to  $-4$ , which points in the positive  $z$  direction. Thus,

$$d\vec{S} = (-2 - 2v/u, 1 - 1/u, 2v + 2) du dv.$$

b. Find an equation for the tangent plane to this surface at the point  $(2, 1, 1)$ ,

$r(u, v) = (2, 1, 1)$  implies that  $u = 1$  and  $v = 1$ . Thus, a normal to the surface at the point  $(2, 1, 1)$  is  $(4, 0, -4)$ . An equation for the tangent plane to the surface is

$$\begin{aligned} (x - 2, y - 1, z - 1) \cdot (-4, 0, 4) &= 0 \\ x - z &= 1. \end{aligned}$$

c. express the area of  $S$  as an iterated integral,

$$\begin{aligned} \text{area} &= \iint_D \|(-2 - 2v/u, 1 - 1/u, 2v + 2)\| du dv \\ &= \int_1^2 du \int_0^2 \|(-2 - 2v/u, 1 - 1/u, 2v + 2)\| du dv. \end{aligned}$$

- d. if  $F(x, y, z) = (x^2, y, z)$ , express the flux of  $\vec{F}$  across the surface  $S$ , with the orientation of  $S$  in the positive  $z$  direction, as an iterated integral.

$$\begin{aligned} \text{flux} &= \iint_S (x^2, y, z) \cdot d\vec{S} \\ &= \iint_D ((u+v)^2, (2u-v^2), v+\ln u) \cdot (-2-2v/u, 1-1/u, 2v+2) du dv \\ &= \int_1^2 du \int_0^2 ((u+v)^2, (2u-v^2), v+\ln u) \cdot (-2-2v/u, 1-1/u, 2v+2) dv \end{aligned}$$

3. (20) Let  $x = u - v^2$  and  $y = u + v$ . Find formulas for  $\frac{\partial u}{\partial x}$  and  $\frac{\partial v}{\partial x}$  in terms of  $u$  and  $v$ .

Differentiating each of the above equations with respect to  $x$  we have

$$1 = u_x - 2vv_x$$

$$0 = u_x + v_x.$$

Solve these equations for  $u_x$  and  $v_x$ .

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{1}{1+2v} \\ \frac{\partial v}{\partial x} &= \frac{-1}{1+2v}. \end{aligned}$$

4. (40) Let  $\vec{F}(x, y) = (x^2y, 4xy + y^2)$  and  $D = \{(x, y) : x^2 + y^2 \leq 4\}$ .

- a. Calculate the circulation of  $\vec{F}$  around the region  $D$  in a counter clockwise direction. That is, calculate the line integral of the tangential component of  $\vec{F}$  in a counter clockwise direction about the region  $D$ .

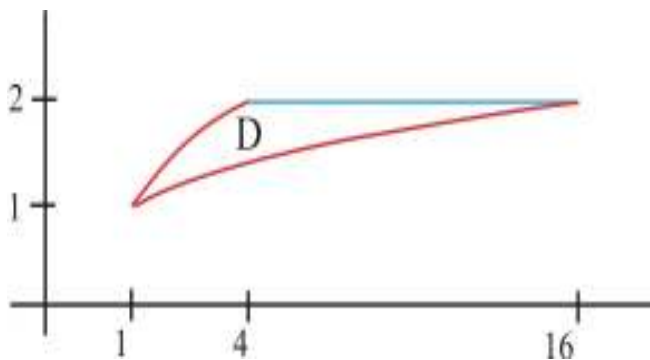
$$\begin{aligned} \text{circulation} &= \int_{\partial D} \vec{F} \cdot d\vec{r} \\ &= \iint_D \left[ \frac{\partial}{\partial x}(4xy + y^2) - \frac{\partial}{\partial y}(x^2y) \right] dA \\ &= \iint_D (4y - x^2) dA \\ &= \int_0^{2\pi} d\theta \int_0^2 (4r \sin \theta - r^2 \cos^2 \theta) r dr \\ &= -\int_0^{2\pi} d\theta \int_0^2 r^3 \cos^2 \theta dr = -\int_0^{2\pi} d\theta \int_0^2 r^3 \frac{1 + \cos 2\theta}{2} dr \\ &= -\pi \int_0^2 r^3 dr = -4\pi \end{aligned}$$

- b. Calculate the outward flux of  $\vec{F}$  across the boundary of  $D$ .

$$\begin{aligned} \text{outward flux} &= \int_{\partial D} \vec{F} \cdot (dy, -dx) = \int_{\partial D} (P, Q) \cdot (dy, -dx) \\ &= \int_{\partial D} (-Q)dx + Pdy \\ &= \iint_D \left[ \frac{\partial P}{\partial x} - \frac{\partial(-Q)}{\partial y} \right] dA \\ &= \iint_D [2xy + 4x + 2y] dA \\ &= \int_0^{2\pi} d\theta \int_0^2 (2r^2 \cos \theta \sin \theta + 4r \cos \theta + 2r \sin \theta) r dr \\ &= 0. \end{aligned}$$

5. (20) The iterated integral  $\int_1^2 dy \int_{y^2}^{y^4} dx$  represents the area of a region  $D$  in  $R^2$ .

a. Sketch the region  $D$ .



b. Express the area of  $D$  as an iterated integral(s) in which the  $y$  variable is integrated first rather than last.

$$\int_1^2 dy \int_{y^2}^{y^4} dx = \int_1^4 dx \int_{x^{1/4}}^{\sqrt{x}} dy + \int_4^{16} dx \int_{x^{1/4}}^2 dy$$