Math 221 - 501 Exam 3 Solutions Nov. 30, 2010

1. (15) Let $\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ be defined on all of R^3, where P, Q, and R are real valued functions. Let $f(x,y,z)$ be a real valued function defined on all of R^3. Assume all four functions have an infinite number of continuous partial derivatives. Let $\vec{\nabla} = (\partial_x, \partial_y, \partial_z)$ denote the gradient operator. Decide whether the following operations make sense. If they do, perform the calculations. If they don't explain why. Note: × denotes cross product, and · denotes the dot product.

   (a) $\vec{\nabla} \times f$

   This does not make sense. f is a scalar, and in order for $\vec{\nabla} \times f$ to make sense it would have to a vector (R^3) valued function.

   (b) $\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{F}\right)$

   This makes sense and equals the zero function.

$$\begin{aligned}\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{F}\right) &= (\partial_x, \partial_y, \partial_z) \cdot (R_y - Q_z, P_z - R_x, Q_x - P_y) \\ &= (R_{yx} - Q_{zx}) + (P_{zy} - R_{xy}) + (Q_{xz} - P_{yz}) = 0\end{aligned}$$

   (c) $\vec{\nabla} f$

   This makes sense and is the gradient of f.

$$\vec{\nabla} f = (\partial_x, \partial_y, \partial_z) f = (f_x, f_y, f_z)\,.$$

2. (25) State each of the following theorems, and prove one of them.

   (a) Fundamental theorem for line integrals.

   Let $\vec{F}$ be a conservative force field whose component functions are continuous. That is, there is a C^1 function Φ such that $\nabla\Phi = \vec{F}$. Then for any C^1 path Γ connecting point P to Q we have:

$$\int_\Gamma \vec{F} \cdot d\Gamma = \Phi(Q) - \Phi(P)\,.$$

   Suppose that $\vec{F} = (P, Q, R)$ and that $\Gamma(t) = (x(t), y(t), z(t))$ for $a \le t \le b$. Then

$$\begin{aligned}\int_\Gamma \vec{F} \cdot d\Gamma &= \int_a^b \left(\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}, \frac{\partial\Phi}{\partial z}\right) \cdot (x', y', z')\,dt \\ &= \int_a^b \frac{d\Phi(x(t), y(t), z(t))}{dt}\,dt = \Phi(x(b), y(b), z(b)) - \Phi(x(a), y(a), z(a)) \\ &= \Phi(Q) - \Phi(P)\end{aligned}$$

(b) Green's theorem, and prove it when the region of integration is a rectangle $[a,b]\times[c,d]$.

Let $\vec{F} = (P(x,y), Q(x,y))$. Let C be a positively oriented piecewise-smooth simple closed curve in the plane. Let D be the region bounded by C. Let P and Q be defined, and have continuous first partial derivatives on an open region containing D. Then

$$\int_C Pdx + Qdy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA\,.$$

Suppose that $D = [a,b] \times [c,d]$. Then we have

$$\begin{aligned}
&\int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\
=\;& \int_a^b dx \int_c^d \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dy = \int_a^b dx \int_c^d \frac{\partial Q}{\partial x} dy - \int_a^b dx \int_c^d \frac{\partial P}{\partial y} dy \\
=\;& \int_c^d \left(Q(b,y) - Q(a,y) \right) dy - \int_a^b \left(P(x,d) - P(x,c) \right) dx \\
=\;& \int_C Qdy + \int_C Pdx = \int_C Pdx + Qdy
\end{aligned}$$

3. (20) Let $\vec{F}(x,y) = (x^2, xy)$. Find the work done by this force field on a particle that moves from $(0,0)$ to $(1,0)$ along the x-axis, and then from $(1,0)$ to $(1,2)$ along the straight line connecting these two points.

The given path can be considered as two paths. The first, C_1 goes from $(0,0)$ to $(1,0)$, and the second, C_2, goes from $(1,0)$ to $(1,2)$. Letting C denote the given path, the work done by this force field is:

$$\begin{aligned}
\text{Work} &= \int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} \\
&= \int_0^1 x^2 dx + \int_0^2 y dy = \frac{1}{3} + 2 = \frac{7}{3}\,.
\end{aligned}$$

4. (20) Let $f(x,y,z) = xy^2z^3$. Let S be the surface described parametrically by $\vec{r}(u,v) = (u^2 - v, u + 2v, 3v - 7u)$ for $1 \le u \le 3$, and $-1 \le v \le 0$. Set up the integral of f over the surface S. You do not have to evaluate your answer, but be sure you clearly show the region of integration and all functions are expressed in terms of the variables of integration.

$$\begin{aligned} \vec{r}_u &= (2u, 1, -7)\,, \quad \vec{r}_v = (-1, 2, 3) \\ \vec{r}_u \times \vec{r}_v &= (17, 7 - 6u, 4u + 1)\,, \quad ||\vec{r}_u \times \vec{r}_v|| = \sqrt{17^2 + (7 - 6u)^2 + (4u + 1)^2} \end{aligned}$$

The surface integral of f is

$$\int\int_S f\, dS = \int_1^3 du \int_{-1}^0 (u^2 - v)(u + 2v)^2(3v - 7u)^3 \sqrt{17^2 + (7 - 6u)2 + (4u + 1)^2}\, dv$$

5. (20) Let $\vec{F}(x,y,z) = (-y, x, z + xy)$. Find the downward flux of the curl of $\vec{F}$ over the top half of the sphere of radius 3 centered at the origin. That is, the surface S is that part of the sphere $x^2 + y^2 + z^2 = 9$, which lies above and on the $z = 0$ plane, and you are to compute $\int\int_S curl\vec{F} \cdot d\vec{S}$, where S has a downward orientation.

In order to use Stokes' theorem we have to paramatrize the the boundary of S, which is the circle of radius 3 in the x-y plane centered at the origin. There are two possible directions, clockwise or counterclockwise. Since we want the downward flux of the curl, we need to have the clockwise orientation. So set $\vec{r}(t) = 3(\sin t, \cos t, 0)$.

$$\int\int_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

$$\begin{aligned} &= \int_0^{2\pi} (-3\cos t, 3\sin t, 9\sin t\cos t) \cdot ((3\cos t, -3\sin t, 0)\, dt \\ &= \int_0^{2\pi} (-9)\, dt = -18\pi\,. \end{aligned}$$