1. (20) Let $f(x,y) = \frac{x^2-y^2}{|x|+|y|}$.

   (a) $\lim_{(x,y)\to(-1,0)} f(x,y) = 1$

   (b) $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. This value becomes clear with the following observation:

$$\left|\frac{x^2-y^2}{|x|+|y|}\right| = \frac{|x-y|\,|x+y|}{|x|+|y|} \le |x-y|\,\frac{|x|+|y|}{|x|+|y|} = |x-y|$$

2. (40) Let $f(x,y,z) = x^2 - 2xy + y^2 - 2z^2$.

   (a) Use the definition of a partial derivative to compute $\frac{\partial f}{\partial z}$ at the point $(1,-1,2)$.

$$\begin{aligned}\frac{\partial f}{\partial z} &= \lim_{h\to 0}\frac{f(1,-1,2+h)-f(1,-1,2)}{h} = \lim_{h\to 0}\frac{(1+2+1-2(2+h)^2)-(-4)}{h}\\ &= \lim_{h\to 0}\frac{-8h-2h^2}{h} = \lim_{h\to 0}(-8-2h) = -8\end{aligned}$$

   (b) Compute the directional derivative of f at the point $(1,-1,2)$ in the direction given by $(3,-2,\sqrt{3})$.

$$\begin{aligned}\mathrm{D}_{\vec{n}} f|_{(1,-1,2)} &= \vec{\nabla} f\Big|_{(1,-1,2)} \cdot \frac{(3,-2,\sqrt{3})}{4}\\ &= (2(x-y), -2(x-y), -4z)|_{(1,-1,2)} \cdot \frac{(3,-2,\sqrt{3})}{4}\\ &= (4,-4,-8)\cdot\frac{(3,-2,\sqrt{3})}{4} = 5 - 2\sqrt{3}\,.\end{aligned}$$

   (c) The function $f(x,y,z)$ when (x,y,z) is restricted to the plane $x+y-6z=1$ does not have a maximum value, but it does have a minimum value. What is this minimum value, and at what point or points does it occur?

   Using Lagrange multipliers with $g(x,y,z) = x+y-6z$, we have

$$\begin{aligned}\vec{\nabla} f + \lambda\vec{\nabla} g &= (2(x-y), -2(x-y)-4z) + \lambda(1,1,-6)\\ 0 &= 2(x-y)+\lambda,\ 0 = -2(x-y)+\lambda,\ \text{these equations imply } \lambda = 0 \text{ and}\\ 0 &= -4z - 6\lambda \text{ implies } z = 0.\end{aligned}$$

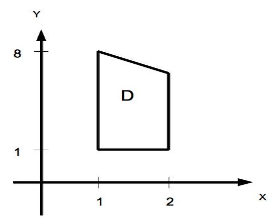
   Thus, we must have $x=y$, and $g(x,x,0)=1$ implies $x = 1/2$. Thus, the minimum value of f on the plane $g=1$ is

$$f(1/2,1/2,0) = 0\,.$$

3. (35) The following iterated integral equals a double integral over a region D in the x-y plane:

$$\int_1^2 dx \int_1^{9-x} dy\,.$$

(a) Sketch the region D.



(b) Find the area of D.

$$\begin{aligned}\text{Area of D} &= \int_1^2 dx \int_1^{9-x} dy = \int_1^2 (8-x)dx = (8x - \frac{x^2}{2})\Big|_1^2 \\ &= \frac{13}{2}\end{aligned}$$

(c) find the x coordinate of the centroid of the region D.

$$\begin{aligned}x_c &= \frac{\int_1^2 dx \int_1^{9-x} x\,dy}{13/2} = \frac{\int_1^2 x(8-x)\,dx}{13/2} = \frac{29/3}{13/2} \\ &= \frac{58}{39}\end{aligned}$$

4. (20) Let $\vec{F}(x, y, z) = (e^{yz} + \cos(x-y), xze^{yz} - \cos(x-y), xye^{yz})$.

(a) Find a potential function ϕ for the vector field F.

From $\partial\Phi/\partial x = e^{yz} + \cos(x-y)$, we get $\Phi = xe^{yz} + \sin(x-y) + g(y, z)$. Checking the other two equations that Φ must satisfy, we see that we can pick $g = 0$.

$$\Phi = xe^{yz} + \sin(x-y)$$

(b) Let $P = (1, 1, \ln 2)$ and $Q = (5, 5, 4)$. Let C be any path that goes from P to Q. Compute $\int_C \vec{F} \cdot d\vec{r}$.

$$\int_C \vec{F} \cdot d\vec{r} = \Phi(5, 5, 4) - \Phi(1, 1, \ln 2) = 5e^{20} - 2\,.$$

5. (35) Let $\vec{F}(x,y,z) = (y^2 z, 2xz^2 + y, x^2 \sin y + z)$, and let E be the region in R^3 bounded above by the plane $z = 0$, and below by the lower half of the sphere $x^2 + y^2 + z^2 = 4$. Find the inward flux of $\vec{F}$ across the boundary of E.

$$\begin{aligned} \text{Inward flux} &= \int\int_{\partial E} \vec{F} \cdot d\vec{S} = -\int\int\int_E \operatorname{div}(\vec{F})\, dV = -\int\int\int_E 2\, dV \\ &= (-2)\text{volume}(E) = (-2)\frac{2}{3}\pi(2^3) = -\frac{32}{3}\pi \end{aligned}$$

Note that the region of integration is half of a sphere of radius 2. Since the volume of a sphere of radius 4 is $\frac{4}{3}\pi r^3$, the volume of this half sphere is $\frac{2}{3}\pi 2^3$.