1. Let $f(x, y, z) = 3x^2 - 2y + z^2$. Let S be the surface which is the locus of points which satisfy the equation $f(x, y, z) = 0$. Find an equation for the plane tangent to S at the point $(1, 2, -1)$.

   Since S is a level surface of the function f, the gradient of f, $\nabla f = (6x, -2, 2z)$ is normal to the surface. Thus, at the point $(1, 2, -1)$ the vector $(6x, -2, 2z)|_{(1,2,-1)} = (6, -2, -2)$ is perpendicular to the surface. An equation to the tangent plane is given by

$$\begin{aligned}(x-1, y-2, z+1) \cdot (6, -2, -2) &= 0 \\ 6x - 4 - 2y - 2z &= 0 \\ 3x - y - z &= 2\end{aligned}$$

2. Find the area of that part of the plane $2x - 3y + 4z = 5$ which is cut out by the cylinder $y^2 + 9z^2 = 4$. Just set up the integral, no need to evaluate it.

   A parametric representation of the surface is $r(y, z) = \left(\frac{5 + 3y - 4z}{2}, y, z\right)$ for (y, z) satisfying $y^2 + 9z^2 \le 4$. The Jacobian of the transformation equals

$$\begin{aligned}J &= \left\| \frac{\partial r}{\partial y} \times \frac{\partial r}{\partial z} \right\| \\ &= \left\| \left(\frac{3}{2}, 1, 0\right) \times (-2, 0, 1) \right\| \\ &= \left\| \left(1, -\frac{3}{2}, 2\right) \right\| \\ &= \left(1 + \left(\frac{3}{2}\right)^2 + 2^2\right)^{1/2} = \sqrt{\frac{29}{4}}\end{aligned}$$

   Thus, the area of the surface is

$$\text{area} = \int_{-2}^{2} dy \int_{-\sqrt{4-y^2}/3}^{\sqrt{4-y^2}/3} \sqrt{\frac{29}{4}}\, dz$$

3. State the following theorems.

   (a) Green's theorem.

   Let R be a bounded region in R^2. Let C denote its boundary and suppose that C can be described as a positively oriented piecewise smooth simple closed curve. If $F(x, y) = (P(x, y), Q(x, y)$ has continuous first partial derivatives in an open region containing R, then

$$\int\int_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int_C P\, dx + Q\, dy$$

(b) Stoke's theorem

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve, C, with positive orientation (right hand rule). Let F be a vector field with continuous first partial derivatives in some open region in R^3 which contains the surface S. Then

$$\iint_S curl\,(F) \cdot dS = \int_C F \cdot d\Gamma$$

(c) The Divergence theorem.

Let Ω be a solid region in R^3 whose boundary surface, $\partial\Omega$, has positive (outward) orientation. Let F be a vector field with continuous first partial derivatives in some open region in R^3 which contains the region Ω. Then

$$\iiint_\Omega div\,(F)\; dV = \iint_{\partial\Omega} F \cdot dS$$

4. Let S be that part of the sphere of radius 2 which is centered at the origin and lies above the plane $z = 1$. Let $F(x, y, z) = (x - y, 2z + 1, x + y)$.

(a) Find a parametric representation for S.

$$r(\theta, \phi) = (2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi)$$
$$\text{for } 0 \le \theta \le 2\pi \text{ and } 0 \le \phi \le \cos^{-1}\left(\frac{1}{2}\right)$$

(b) Find the area of S.

In spherical coordinates we have $dV = \rho^2 \sin\phi d\rho d\phi d\theta$, which means for constant $\rho = 2$ that the surface element $dS = 4\sin\phi d\phi d\theta$. Hence,

$$\begin{aligned} \text{area} &= \int_0^{2\pi} d\theta \int_0^{\cos^{-1}(1/2)} 4\sin\phi\, d\phi \\ &= 8\pi\,(-\cos\phi)\big|_0^{\cos^{-1}(1/2)} \\ &= 4\pi \end{aligned}$$

(c) Use Stokes theorem to find the flux of curl(F) across the surface S.

Stokes theorem implies that the flux of F across the surface S equals the flux of curl(F) across the bottom of the spherical cap. That is that part of the plane $z = 1$ which lies inside the sphere.

$$\begin{aligned} \text{curl}\,(F) &= \det \begin{bmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x - y & 2z + 1 & x + y \end{bmatrix} \\ &= (-1, -1, 1) \end{aligned}$$

Computing the flux of curl(F) across S we have

$$\begin{aligned}
\text{flux} &= \iint_S \operatorname{curl}(F) \cdot dS \\
&= \iint_{z=1} \operatorname{curl}(F) \cdot dS \\
&= \iint_{z=1} (-1,-1,1)\cdot(0,0,1)\ dA \\
&= \iint_{z=1} dA = \text{area of a disk of radius } \sqrt{3} \\
&= 3\pi
\end{aligned}$$

5. Let S be the surface of the preceding problem. Let Ω be the region which is bounded by S and the plane $z=1$. Let $F(x,y,z) = (x-y, 2z+1, x+y)$.

   (a) Find the volume of Ω.

$$\begin{aligned}
\text{volume} &= \int_0^{2\pi} d\theta \int_0^{\sqrt{3}} r\, dr \int_1^{\sqrt{4-r^2}} dz \\
&= 2\pi \int_0^{\sqrt{3}} r\left(\sqrt{4-r^2}-1\right) dr \\
&= \frac{5\pi}{3}
\end{aligned}$$

   (b) Find the outward flux of F across the boundary of Ω.

$$\begin{aligned}
\text{outward flux} &= \iint_{\partial\Omega} F \cdot dS \\
&= \iiint_{\Omega} \operatorname{div}(F)\ dV \\
&= \iiint_{\Omega} \left[\frac{\partial}{\partial x}(x-y) + \frac{\partial}{\partial y}(2z+1) + \frac{\partial}{\partial z}(x+y)\right] dV \\
&= \iiint_{\Omega} dV \\
&= \frac{5\pi}{3}
\end{aligned}$$

6. Let $f(x,y,z) = x^2 + y^2 - 2z$.

(a) Find the rate of change of f at the point $(1,1,1)$ in the direction given by $\vec{n} = (1,3,-5)$.

$$\begin{aligned} D_{\vec{n}} f &= \nabla f|_{(1,1,1)} \cdot \frac{(1,3,-5)}{\sqrt{35}} \\ &= (2x, 2y, -2)|_{(1,1,1)} \cdot \frac{(1,3,-5)}{\sqrt{35}} \\ &= (2,2,-2) \cdot \frac{(1,3,-5)}{\sqrt{35}} \\ &= \frac{18}{\sqrt{35}} \end{aligned}$$

(b) In which direction does f have the greatest rate of change at the point $(1,1,1)$?

The direction for the greatest rate of change of any function is always in the direction of the gradient. In this case, in the direction $(1,1,-1)$.

(c) Find the maximum and minimum values of f on the region $x^2 + y^2 + z^2 \leq 1$.

Since the gradient of f is never zero we know that there are no local extrema in the open set $x^2+y^2+z^2 < 1$. Hence the extreme values of f must occur at some point in the set $x^2+y^2+z^2 = 1$. To find these points Lagrange multipliers are used. Here $g(x,y,z) = x^2 + y^2 + z^2$.

$$\begin{aligned} \nabla f + \lambda \nabla g &= \vec{0} \\ (2x, 2y, -2) + \lambda (2x, 2y, 2z) &= \vec{0} \quad \Rightarrow \\ 2x(1+\lambda) &= 0 \\ 2y(1+\lambda) &= 0 \\ -2(1-\lambda z) &= 0 \end{aligned}$$

There are two cases, $\lambda = -1$ and $\lambda \neq -1$.

For the case $\lambda = -1$: $z = -1$, $x = y = 0$. Which gives us $f(0,0,-1) = 2$.

For the case $\lambda \neq -1$: $x = y = 0$, and $z^2 = 1$. This gives us the two points $(0,0,\pm 1)$. $f(0,0,1) = -2$ and $f(0,0,-1) = 2$.

So the maximum value of f is 2, and the minimum value is -2.