

1. (15) Define the following:

a. the span of the set of vectors $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$,

The set of all linear combinations of these vectors is the span.

b. the set of vectors $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$ is linearly independent.

The set is linearly independent if whenever

$$c_1\vec{x}_1 + \dots + c_k\vec{x}_k = \vec{0}$$

then each of the c_i must equal zero.

c. the null space of an $m \times n$ matrix A .

The null space of A is the set of vectors $\vec{x} \in R^n$ such that $A\vec{x} = \vec{0}$.

2. (25) A system of equations has A as its coefficient matrix, and A is row equivalent to the matrix

$$B = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

a. Find a basis for the null space of A .

The matrix B tells us that

$$x_5 = -x_6, x_2 = x_3 - 2x_6, \text{ and } x_1 = -2x_3 - x_4 - x_6.$$

That is, we have 3 bound variables (x_1, x_2 , and x_5) with the other 3 being free variables. Thus, the null space has dimension 3, and basis equal to

$$\{(-2, 1, 1, 0, 0, 0), (-1, 0, 0, 1, 0, 0), (-1, -2, 0, 0, -1, 1)\}.$$

b. What is the dimension of the column space of A ?

The dimension of the column space is the number of bound variables, which is 3, or, said differently, the number of linearly independent columns.

c. What is the dimension of the row space of A ?

The dimension of the row space is the same as the dimension of the column space. Thus, it too is 3.

d. Explain why there are vectors \vec{b} for which the equation $A\vec{x} = \vec{b}$ does not have a solution? Do you have enough information to exhibit an explicit \vec{b} for which the equation $A\vec{x} = \vec{b}$ does not have a solution?

The vectors \vec{b} lie in R^4 . The dimension of the column space is 3. Since $A\vec{x} = \vec{b}$ is solvable if and only if $\vec{b} \in CS(A)$, and there are vectors in R^4 that do not lie in the three dimensional column space of A , there are vectors \vec{b} for which the equation does not have a solution.

There is not enough information to determine these \vec{b} 's since we don't know the columns of A .

3. (15) Let $A = \begin{bmatrix} 1 & 0 & 6 & 0 \\ 2 & 4 & 2 & 1 \\ 5 & 4 & 1 & 1 \\ 7 & 1 & 0 & 0 \end{bmatrix}$. The determinant of A equals 19, so we know that A has an inverse matrix

B . Find the entry in the second row and third column of B . Hint: $B = \frac{1}{\det(A)} [A_{ij}]^T$, where the A_{ij} are the cofactors of the matrix A .

The entry in the second row and third column of A^{-1} equals

$$\begin{aligned} \frac{1}{\det(A)} A_{3,2} &= \frac{(-1)^{3+2}}{\det(A)} \det \left(\begin{bmatrix} 1 & 6 & 0 \\ 2 & 2 & 1 \\ 7 & 0 & 0 \end{bmatrix} \right) \\ &= \frac{-1}{19} (-1) \det \left(\begin{bmatrix} 1 & 6 \\ 7 & 0 \end{bmatrix} \right) \\ &= \frac{-42}{19}. \end{aligned}$$

4. (20) Let $S = \{ \vec{x} \in R^5 : x_1 + x_3 + x_4 = 0, x_2 + x_4 = 0 \}$.

a. Show that S is a subspace of R^4 .

S is the null space of the matrix $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, so we know that it is a subspace.

b. Find a basis for S .

The equations describing S can be written as

$$x_1 = -x_3 - x_4$$

$$x_2 = -x_4$$

So two free variables, which implies the null space has dimension 2, and a basis is

$$\{(-1, 0, 1, 0), (-1, -1, 0, 1)\}.$$

c. If K is the basis you found in part b. find a basis of R^4 that contains the set K .

We need to find 2 additional vectors. One such pair is $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$. Clearly these 2 vectors along with the two vectors in the basis of S span R^4 . Thus, the set

$$\{(-1, 0, 1, 0), (-1, -1, 0, 1), (1, 0, 0, 0), (0, 1, 0, 0)\}$$

is a basis of R^4 .

d. Find the coordinates of the vector $(4, 5, 1, -5)$ with respect to the basis you found in part c.

We are looking for numbers c_i such that

$$(4, 5, 1, -5) = c_1(-1, 0, 1, 0) + c_2(-1, -1, 0, 1) + c_3(1, 0, 0, 0) + c_4(0, 1, 0, 0).$$

This vector equation leads to the system

$$4 = -c_1 - c_2 + c_3, \quad 5 = -c_2 + c_4$$

$$1 = c_1, \quad -5 = c_2.$$

The solution to this system is $c_1 = 1, c_2 = -5, c_3 = c_4 = 0$. So the coordinates are $[1, -5, 0, 0]$.

5. (15) Let A be an arbitrary $m \times n$ row matrix. Denote the m rows of A by $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_m$ and the n columns of A by $\vec{c}_1, \dots, \vec{c}_n$. Perform the elementary row operation $2R_1 + R_2$ on A and get the matrix B .

a. In terms of the \vec{r}_i what are the rows of B ?

If \vec{b}_i denote the rows of B , then we have

$$\begin{aligned}\vec{b}_i &= \vec{r}_i \text{ for } i \neq 2 \\ \vec{b}_2 &= 2\vec{r}_1 + \vec{r}_2.\end{aligned}$$

b. Show that the span of the rows of A must equal the span of the rows of B .

Since the rows of B are linear combinations of the rows of A it is clear that the span of the rows of A contains the span of the rows of B . To see that any linear combination of the rows of B is in the span of the rows of A , we have

$$\begin{aligned}c_1\vec{b}_1 + c_2\vec{b}_2 + \dots + c_m\vec{b}_m &= c_1\vec{r}_1 + c_2(2\vec{r}_1 + \vec{r}_2) + \dots + c_m\vec{r}_m \\ &= (c_1 + 2c_2)\vec{r}_1 + c_2\vec{r}_2 + \dots + c_m\vec{r}_m.\end{aligned}$$

Thus, any linear combination of the rows of B is also a linear combination of the rows of A .

c. Suppose the first two columns of A are linearly independent. Show that the first two columns of B must be linearly independent.

The first two columns of A and B are respectively

$$\vec{c}_1 = \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1m} \end{bmatrix}, \vec{c}_2 = \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2m} \end{bmatrix}, \vec{B}_1 = \begin{bmatrix} a_{11} \\ 2a_{11} + a_{12} \\ \vdots \\ a_{1m} \end{bmatrix}, \vec{B}_2 = \begin{bmatrix} a_{21} \\ 2a_{21} + a_{22} \\ \vdots \\ a_{2m} \end{bmatrix}$$

where \vec{B}_1 and \vec{B}_2 denote the first two columns of B . Suppose we have

$$\begin{aligned}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} &= \alpha_1\vec{B}_1 + \alpha_2\vec{B}_2 = \alpha_1 \begin{bmatrix} a_{11} \\ 2a_{11} + a_{12} \\ \vdots \\ a_{1m} \end{bmatrix} + \alpha_2 \begin{bmatrix} a_{21} \\ 2a_{21} + a_{22} \\ \vdots \\ a_{2m} \end{bmatrix} \\ &= \alpha_1 \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1m} \end{bmatrix} + \alpha_2 \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2m} \end{bmatrix} + \alpha_1 \begin{bmatrix} 0 \\ 2a_{11} \\ \vdots \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 2a_{21} \\ \vdots \\ 0 \end{bmatrix}\end{aligned}$$

From the first row we see that $\alpha_1 a_{11} + \alpha_2 a_{21} = 0$. This tells us that the sum of the last two terms is $\vec{0}$. Thus, we have the equation

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \alpha_1 \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1m} \end{bmatrix} + \alpha_2 \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2m} \end{bmatrix}.$$

But the first two columns of A are linearly independent. Thus, $\alpha_1 = \alpha_2 = 0$, and we may conclude that the first two columns of B are linearly independent.

6. (10) Suppose $K = \{\vec{x}_1, \dots, \vec{x}_k\}$ is linearly independent and \vec{y} is not in the span of K . Show that the set $\{\vec{x}_1, \dots, \vec{x}_k, \vec{y}\}$ is also linearly independent.

Suppose we have some constants $c_i, 1 \leq i \leq k$, and α such that

$$\vec{0} = c_1\vec{x}_1 + \dots + c_k\vec{x}_k + \alpha\vec{y}.$$

α is either 0 or not. If $\alpha = 0$, then the above equation becomes

$$\vec{0} = c_1\vec{x}_1 + \dots + c_k\vec{x}_k,$$

and from the linear independence of K we know that each of the c_i 's must equal zero. The other possibility is that $\alpha \neq 0$, but then the first equation can be solved for \vec{y}

$$\vec{y} = \left(\frac{-1}{\alpha}\right)(c_1\vec{x}_1 + \dots + c_k\vec{x}_k)$$

From which we have that \vec{y} is a linear combination of the vectors in K . That is \vec{y} is in the span of K . This contradiction says that $\alpha = 0$, and thus, each of the c_i 's is zero. Hence the set $\{\vec{x}_1, \dots, \vec{x}_k, \vec{y}\}$ is linearly independent.