1. (15) Define the following:

a. coordinates of a vector,

If V is a vector space, and $S = \{\vec{u}_1, \cdots, \vec{u}_n\}$ is a basis of V, then the coordinates of a vector $\vec{x} \in V$ with respect to the basis S are the unique scalars c_i, $1 \le i \le n$ such that

$$\vec{x} = \sum_{i=1}^{n} c_i \vec{u}_i \; .$$

b. linear transformation,

A linear transformation, L, from a vector space V to a vector space W is a function with domain V and codomain W, $L : V \to W$, such that

$$L(\alpha\vec{x} + \beta\vec{y}) = \alpha L(\vec{x}) + \beta L(\vec{y}) \; ,$$

for all vectors $\vec{x}$ and $\vec{y}$ in V and any scalars α and β.

c. orthonormal basis.

Let V be a vector space with inner product $\langle \cdot, \; \cdot \rangle$. An orthonormal basis is a basis $\{\vec{u}_1, \cdots, \vec{u}_n\}$ of V such that

$$\langle \vec{u}_i, \vec{u}_j \rangle = \delta_{ij} \, .$$

2. (20) Let $V = P_4$ the vector space of polynomials of degree 3 or less. Let

$$B_1 = \{t^2 - 2t, 1 + t - t^3, 2 - t + t^2, 3t + t^3\},$$
$$B_2 = \{t - t^2 + t^3, 1 + t - t^2, 5t + 7t^2, -6t + 5t^3\}.$$

Find the change of basis matrix P such that

$$[\vec{x}]_{B_1} = P[\vec{x}]_{B_2}.$$

Let $S = \{1, t, t^2, t^3\}$.Let P_1 and P_2 be change of basis matrices such that

$$[\vec{x}]_S = P_1[\vec{x}]_{B_1}$$
$$[\vec{x}]_S = P_2[\vec{x}]_{B2}.$$

Then we must have

$$P_1 = \begin{bmatrix} 0 & 1 & 2 & 0 \\ -2 & 1 & -1 & 3 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad P_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 5 & -6 \\ -1 & -1 & 7 & 0 \\ 1 & 0 & 0 & 5 \end{bmatrix}.$$

To find P, the change of basis matrix we desire we note

$$[\vec{x}]_{B_1} = P_1^{-1}[\vec{x}]_S = P_1^{-1}P_2[\vec{x}]_{B2}.$$

Thus,

$$P = P_1^{-1}P_2$$

$$= \begin{bmatrix} -\frac{4}{7} & \frac{1}{7} & \frac{9}{7} & -\frac{3}{7} \\ -\frac{1}{7} & \frac{2}{7} & \frac{4}{7} & -\frac{6}{7} \\ \frac{4}{7} & -\frac{1}{7} & -\frac{2}{7} & \frac{3}{7} \\ -\frac{1}{7} & \frac{2}{7} & \frac{4}{7} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 5 & -6 \\ -1 & -1 & 7 & 0 \\ 1 & 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{11}{7} & -\frac{12}{7} & \frac{68}{7} & -3 \\ -\frac{8}{7} & -\frac{3}{7} & \frac{38}{7} & -6 \\ \frac{4}{7} & \frac{5}{7} & -\frac{19}{7} & 3 \\ -\frac{1}{7} & -\frac{3}{7} & \frac{38}{7} & -1 \end{bmatrix}.$$

3. (30) Define the following inner product on R^2,

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T A \vec{y},$$

where $A = \begin{bmatrix} 3 & -2 \\ -2 & 5 \end{bmatrix}$.

a. Show that this is an inner product.

The fact that this form is bilinear follows from the fact that matrix multiplication is a linear operation; that it is symmetric follows from the fact that A is a symmetric matrix. The only issue is whether or not the form is positive definite:

$$\begin{aligned}
\langle \vec{x}, \vec{x} \rangle = \vec{x}^T A \vec{x} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 3 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 3x_1 - 2x_2 \\ -2x_1 + 5x_2 \end{bmatrix} \\
&= 3x_1^2 - 4x_1x_2 + 5x_2^2 \\
&= 2x_1^2 + (x_1 - 2x_2)^2 + x_2^2 .
\end{aligned}$$

Thus, if $\langle \vec{x}, \vec{x} \rangle = 0$, we must have $x_1 = x_2 = 0$.

b. Find the distance from the vector $\begin{bmatrix} 5 \\ -1 \end{bmatrix}$ to the subspace $W =$ $\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

The distance is equal to the length of the vector obtained by subtracting the the projection of $\begin{bmatrix} 5, & -1 \end{bmatrix}$ onto W from itself. That is,

$$\begin{aligned}
\text{distance} &= \left\| \begin{bmatrix} 5, & -1 \end{bmatrix} - \text{Proj}_W \begin{bmatrix} 5, & -1 \end{bmatrix} \right\| \\
&= \left\| \begin{bmatrix} 5, & -1 \end{bmatrix} - \frac{-13}{15} \begin{bmatrix} 1, & 2 \end{bmatrix} \right\| \\
&= \left\| \begin{bmatrix} \frac{88}{15}, \frac{11}{15} \end{bmatrix} \right\| \\
&= \frac{11}{15} \left\| \begin{bmatrix} 8, 1 \end{bmatrix} \right\| \\
&= \frac{11}{15} \sqrt{165} \approx 9.420
\end{aligned}$$

4. (20) Find the matrix representation with respect to the standard basis of R^3 of the linear transformation $L : R^3 \to R^3$, which rotates R^3 60° in a counter clockwise direction about the line spanned by the vector $(1, 2, 1)$.

A basis for the plane of rotation is $\{(-2, 1, 0), (-1, 0, 1)\}$. Using Gram-Schmidt we find the following orthonormal basis for this plane:

$$\vec{u}_1 = \frac{1}{\sqrt{5}}(-2, 1, 0) \text{ and } \vec{u}_2 = \frac{1}{\sqrt{30}}(-1, -2, 5).$$

Note that, when viewed from the point $(1, 2, 1)$, $\vec{u}_2$ can be obtained by rotating $\vec{u}_1$ 90° in a counter clockwise direction. Setting $\vec{u}_3 = (1, 2, 1)$, and using these 3 vectors as a basis of R^3, the matrix representation of the linear transformation L with respect to this basis is

$$\hat{A} = \begin{bmatrix} \cos 60 & -\sin 60 & 0 \\ \sin 60 & \cos 60 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If P is the change of basis matrix, which converts coordinates with respect to the $\vec{u}_i$ basis to coordinates with respect to the standard basis of R^3, then

$$P = \begin{bmatrix} -2/\sqrt{5} & -1/\sqrt{30} & 1 \\ 1/\sqrt{5} & -2/\sqrt{30} & 2 \\ 0 & 5/\sqrt{30} & 1 \end{bmatrix},$$

and the matrix representation of L with respect to the standard basis is

$$A = P\hat{A}P^{-1}$$

$$= \begin{bmatrix} -2/\sqrt{5} & -1/\sqrt{30} & 1 \\ 1/\sqrt{5} & -2/\sqrt{30} & 2 \\ 0 & 5/\sqrt{30} & 1 \end{bmatrix} \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2/\sqrt{5} & -1/\sqrt{30} & 1 \\ 1/\sqrt{5} & -2/\sqrt{30} & 2 \\ 0 & 5/\sqrt{30} & 1 \end{bmatrix}^{-1}$$

$$\approx \begin{bmatrix} 0.583 & -0.186 & 0.790 \\ 0.520 & 0.833 & -0.186 \\ -0.623 & 0.520 & 0.583 \end{bmatrix}.$$

5. (15) Let $L : V \to W$ be a linear transformation from the vector space V of dimension n into the vector space W of dimension m. Let $\mathrm{V} = \{\vec{v}_1, \cdots, \vec{v}_n\}$ and $\mathrm{W} = \{\vec{w}_1, \cdots, \vec{w}_m\}$ be bases of V and W respectively. Let A be the matrix representation of L with respect to the bases V and W.

a. Describe the entries of the second column of A.

The entries of the second column of A are the coordinates of $L(\vec{v}_2)$ with respect to the basis W.

b. Show that $\vec{x} \in \ker(L)$ if and only if $[\vec{x}]_{\mathrm{v}} \in \mathrm{NS}(A)$.

The relationship between L and its matrix representation is given by the equation

$$[L(\vec{x})]_{\mathrm{W}} = A[\vec{x}]_{\mathrm{V}}\ .$$

Suppose $\vec{x} \in \ker(L)$, then $L(\vec{x}) = \vec{0}$. Thus, $[L(\vec{x})]_{\mathrm{W}}$ consists of m zeros, which means that $[\vec{x}]_{\mathrm{V}}$ belongs to the null space of A. Conversely, if $[\vec{x}]_{\mathrm{V}}$ belongs to the null space of A, then $[L(\vec{x})]_{\mathrm{W}}$ consists of m zeros, and $L(\vec{x})$ must equal $\vec{0}$, or $\vec{x}$ belongs to the $\ker(L)$.

c. Show that $\vec{y} \in \mathrm{image}(L)$ if and only if $[\vec{y}]_{\mathrm{W}} \in \mathrm{CS}(A)$.

The image of L consists of all vectors in W of the form $L(\vec{x})$. That is, all vectors in W whose coordinates are of the form $[L(\vec{x})]_{\mathrm{W}}$, but this is the same as all vectors in W such that

$$[L(\vec{x})]_{\mathrm{W}} = A[\vec{x}]_{\mathrm{V}}\ .$$

Since $A[\vec{x}]_{\mathrm{V}}$ is a linear combination of the columns of A, we see that $[L(\vec{x})]_{\mathrm{W}}$ lies in the column space of A.