1. (25) Define the following:

a. L is a linear transformation,

L is first of all a function mapping a vector space V into a vector space W with the following additional property: if $\vec{x}$ and $\vec{y}$ are any two vectors in V and α and β are any two scalars, then

$$L(\alpha\vec{x} + \beta\vec{y}) = \alpha L(\vec{x}) + \beta L(\vec{y}) .$$

b. matrix representation of a linear transformation,

Let $L : V \to W$ be a linear transformation. Suppose B_1 and B_2 are bases of V and W respectively. Then A is the matrix representation of L with respect to the given bases if

$$[L(\vec{x})]_{B_2} = A[\vec{x}]_{B_1} .$$

c. linearly independent set of vectors,

A set of vectors $\{\vec{x}_i\}_{i=1}^n$ is linearly independent if whenever

$$c_1\vec{x}_1 + \cdots c_n\vec{x}_n = \vec{0} ,$$

then each of the c_i's must equal 0.

d. eigenvector

If A is an $n \times n$ matrix and $\vec{x} \in R^n$, then $\vec{x}$ is an eigenvector of A if $\vec{x} \neq \vec{0}$ and there is a number λ such that

$$A\vec{x} = \lambda\vec{x} .$$

e. null space of a matrix.

If A is an $m \times n$ matrix, its null space consists of those vectors $\vec{x} \in R^n$ such that

$$A\vec{x} = \vec{0} .$$

2. (30) Let L be the linear transformation the rotates R^3 30° about the line passing through the origin and the point $(-2, 1, 0)$. The direction of rotation is counter clockwise when the origin is viewed from the point $(-2, 1, 0)$.

a. Find the matrix representation of L with respect to the standard basis of R^3.

An orthonormal basis for the plane of rotation is $\vec{u}_1 = (0,0,1)$ and $\vec{u}_2 = (1,2,0)/\sqrt{5}$. Adding the vector $\vec{u}_3 = (-2,1,0)$ we have a basis of R^3. Let $\hat{A}$ be the matrix representation of L with respect to this basis. Then

$$\hat{A} = \begin{bmatrix} \cos 30 & -\sin 30 & 0 \\ \sin 30 & \cos 30 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If P is the change of basis matrix that converts coordinates with respect to the $\vec{u}$'s into coordinates with respect to the standard basis, then

$$P = \begin{bmatrix} 0 & 1/\sqrt{5} & -2 \\ 0 & 2/\sqrt{5} & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Let A denote the matrix representation of L with respect to the standard basis. Then

$$A = P\hat{A}P^{-1}$$

$$= \begin{bmatrix} 0 & 1/\sqrt{5} & -2 \\ 0 & 2/\sqrt{5} & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{5}\sqrt{5} & \frac{2}{5}\sqrt{5} & 0 \\ -\frac{2}{5} & \frac{1}{5} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}+8}{10} & \frac{\sqrt{3}-2}{5} & \frac{\sqrt{5}}{10} \\ \frac{\sqrt{3}-2}{5} & \frac{2\sqrt{3}+1}{5} & \frac{\sqrt{5}}{5} \\ -\frac{\sqrt{5}}{10} & -\frac{\sqrt{5}}{5} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

b. Compute $L(\vec{e}_1)$.

The easiest way to do this is to write $\vec{e}_1$ as a linear combination of the vectors $\vec{u}_i$'s.

$$\vec{e}_1 = \frac{\sqrt{5}}{5}\vec{u}_2 - \frac{2}{5}\vec{u}_3 .$$

Then

$$L(\vec{e}_1) = L\left(\frac{\sqrt{5}}{5}\vec{u}_2 - \frac{2}{5}\vec{u}_3\right) = \frac{\sqrt{5}}{5}L(\vec{u}_2) - \frac{2}{5}L(\vec{u}_3)$$

$$= \frac{\sqrt{5}}{5}\left(-\frac{1}{2}\vec{u}_1 + \frac{\sqrt{3}}{2}\vec{u}_2\right) - \frac{2}{5}\vec{u}_3$$

$$= -\frac{\sqrt{5}}{10}(0,0,1) + \frac{\sqrt{15}}{10}\frac{1}{\sqrt{5}}(1,2,0) - \frac{2}{5}(-2,1,0)$$

$$= \left(\frac{\sqrt{3}+8}{10}, \frac{\sqrt{3}-2}{5}, -\frac{\sqrt{5}}{10}\right).$$

3. (30) Let $A = \begin{bmatrix} -6 & 1 & 0 \\ 1 & 1 & 6 \end{bmatrix}$ be the matrix representation of a linear transformation $L : P_3 \to P_2$ with respect to the standard bases of these vector spaces. That is $\{1, t, t^2\}$ is the basis in P_3 and $\{1, t\}$ is the basis in P_2.

a. Find the kernel of L.

The null space of the matrix A has dimension equal to 1 (rank of $A = 2$), and a basis for the null space is $\{(-6, -36, 7)\}$. Thus, a basis for the ker(L) is $\{-6 - 36t + 7t^2\}$.

b. Find the image of L.

The rank of A is 2, which is the dimension of its column space. So the dimension of the image of L must also be 2. Since the codomain of L is P_2 a vector space of dimension 2, we must have the image of L all of its codomain. That is, $\text{Im}(L) = P_2$.

4. (30) Let $A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$.

a. Find the eigenvalues and eigenvectors of A.

$$\det(A - \lambda I) = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2) .$$

The eigenvalues of A are the roots of this polynomial 3 and -2. The associated eigenvectors are given in the table below

λ	-2	$(1, -4)$
x_λ	3	$(1, 1)$

.

b. Find a diagonal matrix D and a matrix P such that

$$A = PDP^{-1} .$$

$$D = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}$$

c. Calculate A^{99}.

$$\begin{aligned} A^{99} &= PD^{99}P^{-1} \\ &= P\begin{bmatrix} (-2)^{99} & 0 \\ 0 & 3^{99} \end{bmatrix}P^{-1} . \end{aligned}$$

5. (25) Let A be a 3×3 matrix with the following eigenvalues and eigenvectors:

λ	-1	0	2
$\vec{x}_\lambda$	$(1, 1, 0)$	$(0, 1, -1)$	$(2, 0, 1)$

.

a. Show that the eigenvectors $\vec{x}_\lambda$ form a basis for R^3.

The three vectors are linearly independent, R^3 has dimension equal to 3, so they must be a basis.

b. Let $\vec{x} = (1, 3, -2)^{\mathrm{T}}$. Determine $A\vec{x}$.

Write $\vec{x}$ as a linear combination of the given eigenvectors.

$$\vec{x} = (1, 1, 0)^{\mathrm{T}} + 2(0, 1, -1)^{\mathrm{T}}.$$

Then we must have

$$A\vec{x} = A(1, 1, 0)^{\mathrm{T}} + 2A(0, 1, -1)^{\mathrm{T}} = -(1, 1, 0)^{\mathrm{T}} + 2(0)(0, 1, -1)^{\mathrm{T}}$$
$$= -(1, 1, 0)^{\mathrm{T}}.$$