Solutions to Exam 1

1. (20) Define the following terms and give an example of each. NO EXAMPLE, NO CREDIT.

   (a) Subspace

   Let V be a vector space. A subset W of V is a subspace if W is not empty and it is closed under the operations of vector addition and scalar multiplication. An example is : let $V = R^2$ and $W = \{(x, 0) : x \in R\}$.

   (b) Null space of a matrix.

   Let A be an $m \times n$ matrix. The nullspace of A is that set of vectors $\vec{x} \in R^n$ such that $A\vec{x} = \vec{0}$, where $\vec{0}$ is the zero vector in R^m. An example is: let $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$. The null space of A is the subset of R^2, which consists of just the zero vector.

   (c) Column space of a matrix.

   Let A be an $m \times n$ matrix. The nullspace of A is that set of vectors $\vec{x} \in R^m$ which can be written as a linear combination of the columns of A. That is, the column space of a matrix is the span of its column vectors. An example is: let $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, the column space of A is R^2.An example is: let $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$. The null space of A is the subset of R^2An example is: let $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$. The null space of A is the subset of R^2

   (d) Linear independent set of vectors.

   A set of vectors $\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_k\}$ is said to be linearly independent if whenever the equation $c_1\vec{x}_2 + \cdots + c_k\vec{x}_k = \vec{0}$ is true, then all of the constants c_i for $i = 1, 2, \cdots, k$ must equal 0. An example of a linearly independent set is $\{(1, 0), (0, 1)\}$.

   (e) Basis of a vector space.

   A basis of a vector space is a set of linearly independent vectors that also span the vector space. An example of a basis of R^2 is $\{(1, 0), (0, 1)\}$.

2. (10) Is the set W, described below, a subspace of R^3? Be sure to explain your answer.

$$W = \{(x, y, z) : x + 2yz = 0\}, \text{ where } x,\ y, \text{ and } z \text{ are real numbers.}$$

W is not a subspace. It is not closed under either vector addtion or scalar multiplication. To see this notice that $(0, 1, 0)$ and $(0, 0, 1)$ are both in W, but their sum $(0, 1, 1)$ is not in W. To see that W is not closed under scalar multiplication look at the vector $(-2, 1, 1)$. It is in W, but twice it, which equals $(-4, 2, 2)$ is not in W.

3. (30) Consider the following system of linear equations.

$$\begin{aligned} -2x_1 + x_2 - x_3 + 5x_4 &= 1 \\ x_1 + 3x_2 - 4x_3 &= 0 \\ x_1 + x_2 - x_3 - x_4 &= -1 \end{aligned}$$

(a) Find all solutions of this system.

$$\begin{aligned} x_1 &= 2x_4 - \frac{2}{3} \\ x_2 &= -2x_4 - 2 \\ x_3 &= -x_4 - \frac{5}{3} \end{aligned}$$

(b) What is the coefficient matrix, A, of this system?

$$A = \begin{bmatrix} -2 & 1 & -1 & 5 \\ 1 & 3 & -4 & 0 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

(c) Find a basis for the null space of A. What is the dimension of the null space of A?

The matrix A is row equivalent to the matrix $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. This matix has one free variable with the other three bound. Thus, a basis for the null space is $\{(2, -2, -1, 1)\}$, and the dimension of the null space is 1.

(d) Find a basis for the row space of A. What is the dimension of the row space of A?

From part c. we see that the dimension of the row space of A is 3, and that a basis for the row space is $\{(1, 0, 0, -2),\ (0, 1, 0, 2),\ (0, 0, 1, 1)\}$.

4. (30) Let $W = \{A = [a_{ij}] \in M_{2,2} : a_{11} - 3a_{21} = 0\}$. Let $U = \left\{ \begin{bmatrix} 3 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$, and let $V = \left\{ \begin{bmatrix} 6 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \right\}$.

(a) Show that both U and V are bases of W.

We first note that each of the 6 matrices listed above is in the subspace W. To see that U and V are bases, let's first find a basis for W and determine its dimension. Now if $\vec{x}$ is in W we must have

$$\begin{aligned} \vec{x} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 3a_{21} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ &= a_{21} \begin{bmatrix} 3 & 0 \\ 1 & 0 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a_{22} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Thus, we see that the set U above is a spanning set for W. Moreover, it is clear that the three vectors in U are linearly independent. Thus, W has dimension 3 and U is a basis. To see that V is also a basis we note that V contains 3 vectors which are in W and linearly independent, or we can also note that the span of V contains the three vectors in U. Hence the span of V is all of W.

(b) Show that $\vec{x} = \begin{bmatrix} -9 & 5 \\ -3 & 7 \end{bmatrix}$ is a vector in W, and find its coordinates with respect to the basis U.

To see that $\vec{x}$ is in W we check that $a_{11} - 3a_{21} = (-9) - 3(-3) = 0$.

$$\vec{x} = \begin{bmatrix} -9 & 5 \\ -3 & 7 \end{bmatrix} = -3 \begin{bmatrix} 3 & 0 \\ 1 & 0 \end{bmatrix} + 5 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 7 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus, the coordinates of $\vec{x}$ with respect to U are $[-3, 5, 7]$.

(c) Let $\vec{x}$ be a vector in W. If $[\vec{x}]_V = [-2, 5, 3]$, then $\vec{x}$ equals what matrix.

$$\begin{aligned} \vec{x} &= -2 \begin{bmatrix} 6 & 1 \\ 2 & 1 \end{bmatrix} + 5 \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + 3 \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -12 & 3 \\ -4 & 9 \end{bmatrix} \end{aligned}$$

(d) Find the change of basis matrix A such that $[\vec{x}]_V = A\,[\vec{x}]_U$.

We first remark that since W has dimension 3, the matrix A must be a 3×3 matrix. Its first column must give $[\vec{u}_1]_V$, its second column equals $[\vec{u}_2]_V$ and its third column equals $[\vec{u}_3]_V$.

$$\begin{aligned} \vec{u}_1 &= \begin{bmatrix} 3 & 0 \\ 1 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 6 & 1 \\ 2 & 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \\ \vec{u}_2 &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0\begin{bmatrix} 6 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \\ \vec{u}_3 &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0\begin{bmatrix} 6 & 1 \\ 2 & 1 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \end{aligned}$$

Thus, the transition matrix A equals

$$A = \begin{bmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -1/2 & 1/2 \end{bmatrix}$$

5. (10) The matrix $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$ is an elementary row matrix.

(a) What elementary row operation corresponds to E?

-3 times row 3 added to row 2.

(b) Let A be an arbitrary 3×3 matrix. Show that EA performs the elementary operation you found in part **a**. on the matrix A.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then

$$\begin{aligned} EA &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} - 3a_{31} & a_{22} - 3a_{32} & a_{23} - 3a_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{aligned}$$

Now if we add -3 times row 3 of A to row 2 of A we get the same matrix as EA.