Solutions to Exam 2

1. (15) Define the following terms. With each term include an example of what you are defining. No example, no credit.

   (a) Linear transformation from a vector space U into a vector space V.

   A linear transformation, L, from U to V is a function with domain U and co-domain V such that

$$\begin{aligned} L(\vec{x}+\vec{y}) &= L(\vec{x})+L(\vec{y}) \\ L(\alpha\vec{x}) &= \alpha L(\vec{x}) \end{aligned}$$

   An example of a linear transformation from R^2 to R^2 is

$$L(x_1, x_2) = (0, 0).$$

   (b) Length of a vector $\vec{x}$ in R^5.

   If $\vec{x} = (x_1, x_2, x_3, x_4, x_5)$, then $||\vec{x}|| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2}$. A specific example is

$$||(1, 1, 0, 0, 0)|| = \sqrt{2}.$$

   (c) Vector projection of $\vec{x}$ onto $\vec{y}$, where $\vec{x}$ and $\vec{y}$ are two vectors in R^n.

   The projection of $\vec{x}$ onto $\vec{y}$ equals $\dfrac{\vec{x}\cdot\vec{y}}{||\vec{y}||^2}\vec{y}$. An example in R^2 is

$$\text{Proj}_{(1,0)}(3, 4) = \frac{3}{1}(1, 0) = (3, 0).$$

2. (10) Let $\vec{x} = (1, 2, -5, 1)$ and $\vec{y} = (3, -1, 1, 2)$.

   (a) What is the length of $\vec{x}$?

   The length of $\vec{x}$ equals

$$\sqrt{1+4+25+1} = \sqrt{31}.$$

   (b) Are the vectors $\vec{x}$ and $\vec{y}$ perpendicular?

   The dot product of $\vec{x}$ and $\vec{y}$ equals $3 - 2 - 5 + 2 = -2$. Since the dot product is not equal to zero, the vectors are not perpendicular.

3. (25) Let $V = \{(x_1, x_2, x_3, x_4) : x_1 + x_2 + x_4 = 0\}$.

(a) Find a basis of V.

The coefficient matrix of the system of equations $x_1 + x_2 + x_4 = 0$ is

$$\begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}$$

The free variables are x_2, x_3, and x_4. Thus, a basis of V is

$$\{(-1, 1, 0, 0)\,,\ (0, 0, 1, 0)\,,\ (-1, 0, 0, 1)\}\,.$$

(b) Find an orthonormal basis of V.

Call the vectors in the basis from part a. $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ respectively. Then an orthonormal basis of V is

$$\begin{aligned}
\vec{u}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{2}}(-1, 1, 0, 0) \\
\vec{u}_2 &= \frac{\vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\,\vec{u}_1}{\|\vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\,\vec{u}_1\|} = \frac{(0, 0, 1, 0) - 0\vec{u}_1}{\|\vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\,\vec{u}_1\|} = (0, 0, 1, 0) \\
\vec{u}_3 &= \frac{\vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\,\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\,\vec{u}_2}{\|\vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\,\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\,\vec{u}_2\|} \\
&= \frac{\vec{v}_3 - \frac{1}{\sqrt{2}}\vec{u}_1 - 0\vec{u}_2}{\|\vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\,\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\,\vec{u}_2\|} = \frac{1}{\sqrt{6}}(-1, -1, 0, 2)
\end{aligned}$$

(c) Find the projection of $\vec{x} = (1, 1, 1, 1)$ onto V.

The projection of $\vec{x}$ onto V equals

$$\begin{aligned}
\mathrm{Proj}_V\,\vec{x} &= (\vec{x} \cdot \vec{u}_1)\,u_1 + (\vec{x} \cdot \vec{u}_2)\,u_2 + (\vec{x} \cdot \vec{u}_3)\,u_3 \\
&= 0\,\vec{u}_1 + \vec{u}_2 + 0\,\vec{u}_3 = \vec{u}_2 \\
&= (0, 0, 1, 0)
\end{aligned}$$

(d) Find the distance from $\vec{x} = (1, 1, 1, 1)$ to V.

The distance from $\vec{x}$ to V equals

$$\begin{aligned}
\|\vec{x} - \mathrm{Proj}_V\,\vec{x}\| &= \|(1, 1, 1, 1) - (0, 0, 1, 0)\| \\
&= \|(1, 1, 0, 1)\| \\
&= \sqrt{3}\,.
\end{aligned}$$

4. (20) Let $L : P_3 \to M_{2,2}$ be defined by

$$L(p) = \begin{bmatrix} p(0) & p(2) \\ p(0)+p(2) & \int_0^2 p(x)\,dx \end{bmatrix}.$$

(a) Compute $L(1)$.

$$L(1) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

(b) Find the matrix representation of L with respect to the standard bases of P_3 and $M_{2,2}$. These bases are $\{1, t, t^2\}$ and $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ respectively.

$$L(1) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, L(t) = \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}, L(t^2) = \begin{bmatrix} 0 & 4 \\ 4 & 8/3 \end{bmatrix}.$$

The coordinates of these vectors with respect to the standard basis in $M_{2,2}$ are

$$\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 4 \\ 8/3 \end{bmatrix}$$

Thus, the matrix representation of L with respect to the standard bases is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 4 \\ 2 & 2 & 4 \\ 2 & 2 & 8/3 \end{bmatrix}.$$

5. (15) Find an equation for the straight line which best fits the set of points $\{(-2,2)\,,\ (-1,0)\,,\ (1,1)\,,\ (3,2)\}$.

If $y = mx + b$ is the equation of the line, the coefficients m and b should satisfy the system of equations

$$\begin{aligned} -2m + b &= 2 \\ -m + b &= 0 \\ m + b &= 1 \\ 3m + b &= 2 \end{aligned}$$

Writing this as a matrix equation we have $A\vec{x} = \vec{y}$ where

$$A = \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}, \text{ and } \vec{y} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

This system does not have a solution, so the normal equations are used to find the least squares solution

$$\begin{aligned} \vec{x} &= \left(A^T A\right)^{-1} A^T \vec{y} \\ &= \begin{bmatrix} 15 & 1 \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} -2 & -1 & 1 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 4/59 & -1/59 \\ -1/59 & 15/59 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 7/59 \\ 72/59 \end{bmatrix}. \end{aligned}$$

Thus, the best straight line fit to the data is

$$y = \frac{7}{59}x + \frac{72}{59} \approx 0.119x + 1.22\,.$$

6. (15) Suppose that L is a linear transformation from R^2 into R^2. Let $\vec{v}_1 = (1, -3)$ and $\vec{v}_2 = (3, 1)$. Suppose that $L(\vec{v}_1) = \vec{v}_1$ and $L(\vec{v}_2) = -\vec{v}_2$.

(a) Describe this transformation geometrically.

The vectors $\vec{v}_1$ and $\vec{v}_2$ are orthogonal and hence are linearly independent. They therefore form a basis of R^2. Since L leaves $\vec{v}_1$ fixed and sends $\vec{v}_2$ to $-\vec{v}_2$, it is clear that L reflects R^2 through the straight line passing through the origin in the direction of $\vec{v}_1$. For if $\vec{x}$ is any vector in R^2, we write $\vec{x} = a_1\vec{v}_1 + a_2\vec{v}_2$. Thus,

$$\begin{aligned} L(\vec{x}) &= a_1 L(\vec{v}_1) + a_2 L(\vec{v}_2) \\ &= a_1\vec{v}_1 - a_2\vec{v}_2\,. \end{aligned}$$

(b) Find the matrix representation of L with respect to the standard basis of R^2.

We first need to write $\vec{e}_1$ and $\vec{e}_2$ as linear combinations of $\vec{v}_1$ and $\vec{v}_2$. The matrix

$$P = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$$

is the transition (change of basis) matrix from the basis $\{\vec{v}_1, \vec{v}_2\}$ to the standard basis. Thus, the columns of P^{-1} will be the coordinates of the standard basis with respect to the basis $\{\vec{v}_1, \vec{v}_2\}$.

$$P^{-1} = \frac{1}{10}\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$$

Thus,

$$\begin{aligned} L(\vec{e}_1) &= \frac{1}{10}L(\vec{v}_1) + \frac{3}{10}L(\vec{v}_2) = \frac{1}{10}\vec{v}_1 - \frac{3}{10}\vec{v}_2 = \frac{1}{10}(-8, -6) \\ L(\vec{e}_2) &= \frac{-3}{10}L(\vec{v}_1) + \frac{1}{10}L(\vec{v}_2) = \frac{-3}{10}\vec{v}_1 - \frac{1}{10}\vec{v}_2 = \frac{1}{10}(-6, 8) \end{aligned}$$

Thus, the matrix representation of L with respect to the standard basis of R^2 is

$$\begin{bmatrix} -8/10 & -6/10 \\ -6/10 & 8/10 \end{bmatrix} = \begin{bmatrix} -4/5 & -3/5 \\ -3/5 & 4/5 \end{bmatrix}.$$