1. (15) Define each of the following. You must give an example of each term you define.

   (a) Linear transformation

   A linear transformation is a mapping $L$ from a vector space $V$ into a vector space $U$ such that

$$\begin{aligned} L(\vec{x}+\vec{y}) &= L(\vec{x})+L(\vec{y}), \text{ for all } \vec{x} \text{ and } \vec{y} \text{ in } V, \\ L(\alpha\vec{x}) &= \alpha L(\vec{x}) \text{ for all scalars } \alpha \text{ and } \vec{x} \text{ in } V. \end{aligned}$$

   An example is $L(x,y)=x$. so $L$ is a linear transformation from $R^2$ to $R$.

   (b) Eigenvalue

   An eigenvalue of a matrix $A$ is a number $\lambda$ for which there is a non-zero vector $x$ such that

$$A\vec{x}=\lambda\vec{x}.$$

   If $A=\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$, then the eigenvalues of $A$ are 9 and 1.

   (c) Null space of a matrix.

   Let $A$ be an $m\times n$ matrix. The null space of $A$ is the set of $\vec{x}\in R^n$ such that $A\vec{x}=\vec{0}_m$. If $A=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, the null space of $A$ is the span of $(1,0)$.

2. (10) Let $A$ be an $n\times n$ matrix. Suppose $\lambda_1$ and $\lambda_2$ are two different eigenvalues of $A$, and that $\vec{x}_1$ and $\vec{x}_2$ are eigenvectors of $\lambda_1$ and $\lambda_2$ respectively. Show that the set $\{\vec{x}_1, \vec{x}_2\}$ is linearly independent in $R^n$.

   Suppose $c_1\vec{x}_1+c_2\vec{x}_2=\vec{0}$. Then multiplying this equation by $A$ we get the following system

$$\begin{aligned} c_1\vec{x}_1+c_2\vec{x}_2 &= \vec{0} \\ c_1\lambda_1\vec{x}_1+c_2\lambda_2\vec{x}_2 &= \vec{0} \end{aligned}$$

   Multiply the first equation by $\lambda_1$, subtract it from the second equation, and get

$$c_2(\lambda_2-\lambda_1)\vec{x}_2=\vec{0}$$

   Since both $\vec{x}_2$ and $(\lambda_2-\lambda_1)$ are not zero we must have $c_2=0$. This implies that $c_1=0$. Thus, the vectors $\vec{x}_1$ and $\vec{x}_2$ are linearly independent.

3. (15) Let $A$ be a $2 \times 2$ matrix, which has the following eigenvalues and associated eigenvectors.

$$\begin{aligned} \lambda_1 &= \frac{1}{3}, \quad \vec{v}_1 = (-2, 1) \\ \lambda_2 &= 1, \quad \vec{v}_2 = (1, 1) \end{aligned}$$

(a) Let $\vec{x} = (2, 1)$. Compute $A^2\, \vec{x}$.

First write $\vec{x}$ in terms of the eigenvectors.

$$\begin{aligned} \vec{x} &= (2,1) = -\frac{1}{3}\vec{v}_1 + \frac{4}{3}\vec{v}_2. \text{ Thus,} \\ A^2\vec{x} &= -\frac{1}{3}A^2\vec{v}_1 + \frac{4}{3}A^2\vec{v}_2 \\ &= -\frac{1}{3}\left(\frac{1}{3}\right)^2 \vec{v}_1 + \frac{4}{3}\vec{v}_2 \\ &= -\frac{1}{27}(-2,1) + \frac{4}{3}(1,1) \\ &= \left(\frac{38}{27}, \frac{35}{27}\right) \end{aligned}$$

(b) Let $\vec{x} = (2, 1)$. Compute $\lim_{n \to \infty} A^n\, \vec{x}$.

$$\begin{aligned} \lim_{n\to\infty} A^n\,\vec{x} &= \lim_{n\to\infty} A^n\left(-\frac{1}{3}\vec{v}_1 + \frac{4}{3}\vec{v}_2\right) \\ &= \lim_{n\to\infty}\left(-\frac{1}{3}A^n\vec{v}_1 + \frac{4}{3}A^n\vec{v}_2\right) \\ &= \lim_{n\to\infty}\left(-\frac{1}{3}\left(\frac{1}{3}\right)^n \vec{v}_1 + \frac{4}{3}\vec{v}_2\right) \\ &= \frac{4}{3}\vec{v}_2 = \left(\frac{4}{3}, \frac{4}{3}\right) \end{aligned}$$

This problem can also be solved by determing the matrix $A$ from the formula $A = PDP^{-1}$, where

$$P = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and, } D = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then we have $A^n = PD^nP^{-1} = P\begin{bmatrix} (1/3)^n & 0 \\ 0 & 1 \end{bmatrix}P^{-1}$.

4. (30) Let $A = \begin{bmatrix} 9 & -7 & 7 \\ 1 & 1 & 7 \\ 1 & -1 & 9 \end{bmatrix}$.

(a) Find the eigenvalues and eigenvectors of $A$. Hint: the numbers are not complicated, so if you start getting messy answers look for an algebraic mistake.

$$\begin{aligned} \det(A - \lambda I) &= \det\left(\begin{bmatrix} 9-\lambda & -7 & 7 \\ 1 & 1-\lambda & 7 \\ 1 & -1 & 9-\lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} 9-\lambda & 2-\lambda & 7 \\ 1 & 2-\lambda & 7 \\ 1 & 0 & 9-\lambda \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 8-\lambda & 0 & 0 \\ 1 & 2-\lambda & 7 \\ 1 & 0 & 9-\lambda \end{bmatrix}\right) = (8-\lambda)\det\left(\begin{bmatrix} 2-\lambda & 7 \\ 0 & 9-\lambda \end{bmatrix}\right) \\ &= (8-\lambda)(2-\lambda)(9-\lambda) \end{aligned}$$

Thus, the eigenvalues are 8, 2, and 9. The eigenvectors of these eigenvalues are calculated below.

$\lambda = 8$

$$A - 8I = \begin{bmatrix} 1 & -7 & 7 \\ 1 & -7 & 7 \\ 1 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

An eigenvector associated with $\lambda = 8$ is $(0, 1, 1)$.

$\lambda = 2$

$$A - 2I = \begin{bmatrix} 7 & -7 & 7 \\ 1 & -1 & 7 \\ 1 & -1 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

An eigenvector associated with $\lambda = 2$ is $(1, 1, 0)$.

$\lambda = 9$

$$A - 9I = \begin{bmatrix} 0 & -7 & 7 \\ 1 & -8 & 7 \\ 1 & -1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

An eigenvector associated with $\lambda = 9$ is $(1, 1, 1)$.

(b) Find matrices $D$ and $P$, where $P$ is non-singular, such that

$$A = PDP^{-1}.$$

Two possibilities for $P$ and $D$ are:

$$P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

(c) Find a matrix $B$ such that $B^5 = A$.

$$\begin{aligned} B &= A^{1/5} = PD^{1/5}P^{-1} \\ &= P \begin{bmatrix} 8^{1/5} & 0 & 0 \\ 0 & 2^{1/5} & 0 \\ 0 & 0 & 9^{1/5} \end{bmatrix} P^{-1} \\ &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8^{1/5} & 0 & 0 \\ 0 & 2^{1/5} & 0 \\ 0 & 0 & 9^{1/5} \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \\ &\approx \begin{bmatrix} 1.551 & -0.403 & 0.403 \\ 0.0361 & 1.112 & 0.403 \\ 0.0361 & -0.0361 & 1.552 \end{bmatrix} \end{aligned}$$

5. (10) Suppose that $A$ is a $5 \times 7$ matrix with real entries and rank equal to 4.

(a) $A$ can be thought of as a linear transformation from $R^m$ to $R^n$. What are the values of $m$ and $n$?

$m = 7$ and $n = 5$.

(b) What is the dimension of the null space of $A$.

The dimension of the null space is

$$\begin{aligned} \dim(\text{nullspace}) &= \dim(R^m) - \text{rank}(A) \\ &= 7 - 4 = 3 \end{aligned}$$

(c) Must the equation $A\vec{x} = \vec{b}$ always have a solution for any $\vec{b} \in R^n$?

The equation will always have a solution if and only if the dimension of the column space of $A$ equals $n$. In this case we must have $4 = 5$, which is certainly not true. Thus, there is a $\vec{b}$ for which the equation $A\vec{x} = \vec{b}$ does not have a solution.

6. (10) Let $U = \{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ be a basis for $P_4$. Let $P$ be the transition (change of basis) matrix from the basis $U$ to the standard basis of $P_4$. That is,

$$[\vec{x}]_{St} = P\,[\vec{x}]_U\,, \text{ where } P = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 2 \\ 1 & 2 & 1 & 0 \end{bmatrix}$$

Remember, the standard basis of $P_4$ is $\left\{1, t, t^2, t^3\right\}$.

(a) What is the vector $\vec{u}_2$?

The coordinates of $\vec{u}_2$ with respect to the standard basis form the second column of $P$. Thus, $\vec{u}_2 = t + 2t^3$.

(b) What are the coordinates of $t^3$ with respect to the basis $U$?

We need to find $a$, $b$, $c$, and $d$ such that

$$\begin{aligned} t^3 &= a\vec{u}_1 + b\vec{u}_2 + c\vec{u}_3 + d\vec{u}_4 \\ &= a\left(1 + t^2 + t^3\right) + b\left(t + 2t^3\right) + c\left(1 + t^3\right) + d\left(1 - t + 2t^2\right) \\ &= (a + c + d) + (b - d)\,t + (a + 2d)\,t^2 + (a + 2b + c)\,t^3 \end{aligned}$$

This leads to the system of equations

$$\begin{aligned} a + c + d &= 0 \\ b - d &= 0 \\ a + 2d &= 0 \\ a + 2b + c &= 1 \end{aligned}$$

The solution to this system is $a = -2$, $b = 1$, $c = 1$, and $d = 1$. Thus,

$$\left[t^3\right]_U = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

We could also have computed the inverse of $P$. That is,

$$P^{-1} = \begin{bmatrix} 2 & 4 & 1 & -2 \\ -1 & -1 & 0 & 1 \\ 0 & -2 & -1 & 1 \\ -1 & -2 & 0 & 1 \end{bmatrix}$$

The last column of $P^{-1}$ contains the coordinates of $t^3$ with respect to $U$.

7. (10) Let $L : P_3 \to M_{2,1}$ be a linear transformation such that

$$L\left(p\right) = \begin{bmatrix} p\left(1\right) - p\left(2\right) \\ \int_{-1}^{1} p(t)\,dt \end{bmatrix}$$

(a) Find the kernel of $L$.

The kernel of $L$ consists of all $p$ such that $L\left(p\right) = \vec{0}$. Thus, if $p\left(t\right) = a + bt + ct^2$, then

$$\begin{aligned} p\left(1\right) - p\left(2\right) &= \left(a+b+c\right) - \left(a+2b+4c\right) = 0 \\ \int_{-1}^{1} p\left(t\right)\,dt &= 2a + \frac{2c}{3} = 0 \end{aligned}$$

The solution set for this system is spanned by $\left(-\frac{1}{3}, -3, 1\right)$. Thus, the kernel of $L$ is the span of the vector $-\frac{1}{3} - 3t + t^2$.

(b) Find the image of $L$.

Since the dimension of the kernel of $L$ is 1, we know that the dimension of the image of $L$ is $3 - 1 = 2$. Thus, the dimension of the image of $L$ is the same as the dimension of the co-domain of $L$, or the image of $L$ is all of $M_{2,1}$.