1. (30) Let $T : V \to V$ be a linear transformation. Let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a basis of V. Suppose the matrix representation A of T with respect to this basis is

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -4 \\ -2 & -5 & 0 \end{bmatrix}.$$

(a) In terms of the basis vectors what does $T(\vec{v}_1)$ equal?

The first column of A contains the coordinates of $T(\vec{v}_1)$, so

$$T(\vec{v}_1) = \vec{v}_1 - 2\vec{v}_3$$

(b) Does the equation $T(\vec{x}) = \vec{v}_2 - \vec{v}_3$ have a solution? If yes, find it, if no why not.

In terms of coordinates the question is: does there exist an $\vec{y} \in R_3$ such that

$$A\vec{y} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

The augmented matrix for this system is

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -4 & 1 \\ -2 & -5 & 0 & -1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Thus, $\vec{y} = [-2, 1, 0]$, and the solution to the original equation is

$$\vec{x} = -2\vec{v}_1 + \vec{v}_2$$

2. (50) A linear transformation $E : V \to V$ is called a projection if it satisfies the equation $E^2 = E$.

(a) Let $V = R^3$. The set $U = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\} = \{\langle 1, 1, 0\rangle, \langle 1, 0, 1\rangle, \langle 0, 1, 1\rangle\}$ is a basis of R^3. Define E by $E(\lambda_1\vec{u}_1 + \lambda_2\vec{u}_2 + \lambda_3\vec{u}_3) = \lambda_1\vec{u}_1$. Show that E is a projection. Note, this means you need to show that E is a linear transformation, and that $E^2 = E$.

Let $\vec{x} = \lambda_1\vec{u}_1 + \lambda_2\vec{u}_2 + \lambda_3\vec{u}_3$ and $\vec{y} = \mu_1\vec{u}_1 + \mu_2\vec{u}_2 + \mu_3\vec{u}_3$. Then,

$$\begin{aligned} E(\vec{x} + \vec{y}) &= E((\lambda_1 + \mu_1)\vec{u}_1 + (\lambda_2 + \mu_2)\vec{u}_2 + (\lambda_3 + \mu_3)\vec{u}_3) \\ &= (\lambda_1 + \mu_1)\vec{u}_1 = E(\vec{x}) + E(\vec{y}) \end{aligned}$$

$$\begin{aligned} E(\alpha\vec{x}) &= E(\alpha(\lambda_1\vec{u}_1 + \lambda_2\vec{u}_2 + \lambda_3\vec{u}_3)) = E((\alpha\lambda_1)\vec{u}_1 + (\alpha\lambda_2)\vec{u}_2 + (\alpha\lambda_3)\vec{u}_3) \\ &= (\alpha\lambda_1)\vec{u}_1 = \alpha(\lambda_1\vec{u}_1) = \alpha E(\vec{x}). \end{aligned}$$

Thus, E is a linear transformation. To see that it is also a projection we have

$$E^2(\vec{x}) = E(E(\vec{x}_1)) = E(\lambda_1\vec{u}_1) = \lambda_1\vec{u}_1 = E(\vec{x}).$$

(b) What is the matrix representation of E with respect to the basis U ?

From $E(\vec{u}_1) = \vec{u}_1$, and $E(\vec{u}_2) = E(\vec{u}_3) = \vec{0}$, we have

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) Let S denote the standard basis of R_3. What is the change of basis matrix P for which

$$[\vec{x}]_S = P[\vec{x}]_U \ ?$$

From the equation defining P we see that the columns of P must be the coordinates of the vectors $\vec{u}_i$ with respect to the standard basis S. Thus,

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

(d) What is the matrix representation of E with respect to the basis S ?

The matrix, call it B, satisfies the equation $[E\vec{x}]_S = B[\vec{x}]_S$. So

$$\begin{aligned} [E\vec{x}]_S &= P[E\vec{x}]_U = PA[\vec{x}]_U \\ &= PAP^{-1}[\vec{x}]_s. \end{aligned}$$

So we have

$$\begin{aligned} B &= PAP^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

3. (20) Let $T : V \to W$ be a linear transformation.

(a) For any subspace S of V show that $T(S)$ is subspace of W.

Let $\vec{u}$ and $\vec{v}$ belong to $T(S)$. Then there exist $\vec{x}$ and $\vec{y}$ in S such that $T(\vec{x}) = \vec{u}$ and $T(\vec{y}) = \vec{v}$. Then we have

$$\begin{aligned} \vec{u} + \vec{v} &= T(\vec{x}) + T(\vec{y}) = T(\vec{x} + \vec{y}) \in T(S) \\ \alpha\vec{u} &= \alpha T(\vec{x}) = T(\alpha\vec{x}) \in T(S). \end{aligned}$$

Thus, $T(S)$ is closed under vector addition and scalar multiplication, and since $T\left(\vec{0}\right) \in T(S)$, $T(S)$ is a subspace of W.

(b) For any subspace Q of W show that $T^{-1}(Q)$ is a subspace of V.

Let $\vec{x}$ and $\vec{y}$ belong to $T^{-1}(Q)$. That is $T(\vec{x})$ and $T(\vec{y})$ belong to Q. Thus, since Q is a subspace we have

$$T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y}) \in Q,$$

this is the statement that $\vec{x}+\vec{y}$ belongs to $T^{-1}(Q)$. We also have

$$T(\alpha\vec{x}) = \alpha T(\vec{x}) \in Q.$$

Thus, $T^{-1}(Q)$ is also closed under scalar multiplication and $\vec{0} \in T^{-1}(Q)$. So $T^{-1}(Q)$ is a subspace.