1. (15) What shall I name my pond snake?

   The top three names are:

   Eigen from Ashley Hubble
   Pond, James Pond agent 008 from Amy Hopson
   Sam, aka 'Snake at Mike's' from Valerie Berlin

2. (25) Let $T : V \to V$ be a linear transformation from V into V, where V is assumed to be a finite dimensional vector space over an algebraically closed field F.

   (a) Suppose M is an invariant subspace under T. Show that M is invariant under $T - \lambda I$, for any scalar λ, where I is the identity transformation from V into V. Conversely show that if M is invariant under $T - \lambda I$, for a particular scalar λ, then M is invariant under T.

   Since M is an invariant subspace for any $\vec{x} \in M$ we have $\lambda \vec{x} \in M$ and $T\vec{x} \in M$, and therefore $T\vec{x} - \lambda \vec{x} \in M$. Thus, M is invariant under $T - \lambda I$, for any $\lambda \in F$. If M is invariant under $T - \lambda I$, then by the first part of this problem we have M invariant under $(T - \lambda I) - (-\lambda) I = T$, and M is invariant under T.

   (b) Let c be an eigenvalue of T. Show that ac is an eigenvalue of the linear transformation aT, and in general if $f(t) = a_0 + a_1 t + \cdots + a_n t^n$ is any polynomial with coefficients $a_i \in F$, then $f(c)$ is an eigenvalue of the linear transformation

   $$f(T) = a_0 I + a_1 T + \cdots + a_n T^{\,n}.$$

   Let $\vec{x}$ be any eigenvector associated with the eigenvalue c. That is, $T\vec{x} = c\vec{x}$. This equation implies

   $$\begin{aligned} T^n \vec{x} &= T^{n-1} T \vec{x} = T^{n-1}(c\vec{x}) = cT^{n-1}\vec{x} \\ &= c\left(c^{n-1}\vec{x}\right) = c^n \vec{x}. \end{aligned}$$

   We also have

   $$(aT)(\vec{x}) = a(c\vec{x}) = (ac)\vec{x}.$$

   In summary ac is an eigenvalue of aT and c^n is an eigenvalue of T^n, and any eigenvector of T associated with the eigenvalue c is an eigenvector of aT and T^n associated with the eigenvalues ac and c^n respectively. Thus, we have

   $$\begin{aligned} f(T)(\vec{x}) &= \left(a_0 I + a_1 T + \cdots + a_n T^{\,n}\right)(\vec{x}) \\ &= a_0 I(\vec{x}) + (a_1 T)(\vec{x}) + \cdots + \left(a_n T^{\,n}\right)(\vec{x}) \\ &= a_0 \vec{x} + a_1 (c\vec{x}) + \cdots + a_n (c^n \vec{x}) \\ &= (a_0 + a_1 c + \cdots + a_n c^n)(\vec{x}) \\ &= f(c)\,\vec{x}\,. \end{aligned}$$

   Hence $f(c)$ is an eigenvalue of $f(T)$ with associated eigenvector $\vec{x}$.

(c) Let c be an eigenvalue of T, and let E_c be the generalized eigenspace of T associated with the eigenvalue c. Let T_c denote the linear transformation T restricted to E_c. That is,

$$T_c : E_c \to E_c,$$

where $T_c(\vec{x}) = T(\vec{x})$.
Show that T_c has only one eigenvalue, and that it is c. Deduce from this that the characteristic polymomial of T_c equals $p(\lambda) = (\lambda - c)^m$, where $m = \dim(E_c)$.

Let q be any eigenvalue of T_c with an associated eigenvector $\vec{x}$. Since $\vec{x} \in E_c$ there is an integer k such that $(T - cI)^k \vec{x} = \vec{0}$. But $\vec{x}$ is an eigenvector associated with the eiqenvalue q. So from part b. and this equation we have

$$\vec{0} = (T - cI)^k \vec{x} = (q - c)^k \vec{x}\ .$$

Since $\vec{x} \neq \vec{0}$, we must have $q = c$. Thus, the only eigenvalue of T_c is c. Moreover since we're assuming that the field F is algebraically closed we know that the characteristic polynomial of T_c can be factored into linear factors. That is,

$$\det(T_c - \lambda I) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_k)^{m_k}\ ,$$

where the λ_i's are the distinct eigenvalues. Since T_c has only one eigenvalue we have

$$\det(T_c - \lambda I) = (\lambda - c)^{\dim(E_c)}\ .$$

3. (30) Let $T : R_4 \to R_4$ be a linear transformation. Suppose the matrix representation of T with respect to the basis $B = \{\vec{v}_i\}_{i=1}^4$ is

$$J = \begin{bmatrix} 1/2 & 1 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

That is $T(\vec{v}_1) = (1/2)\vec{v}_1$, etc.

(a) What are the eigenspaces and generalized eigenspaces of T. Your answer should be expressed in terms of the basis vectors in B.

From J we see that

$$\begin{aligned} \det(J - \lambda I) &= \det \begin{bmatrix} 1/2-\lambda & 1 & 0 & 0 \\ 0 & 1/2-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 1 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix} \\ &= (\lambda - 1/2)^2 (\lambda - 1)^2 \\ T(v_1) &= \frac{1}{2}v_1, \quad T(v_2) = \frac{1}{2}v_2 + v_1 \\ T(v_3) &= v_3, \quad T(v_4) = v_4 + v_3 \end{aligned}$$

Another way to write these equations is as follows:

$$\begin{aligned}\left(T-\frac{1}{2}I\right)v_1 &= 0,\ \left(T-\frac{1}{2}I\right)v_2 = v_1 \implies \left(T-\frac{1}{2}I\right)^2 v_2 = 0\\ (T-I)\,v_3 &= 0,\ (T-I)\,v_4 = v_3 \implies (T-I)^2\,v_4 = 0.\end{aligned}$$

So, the eigenvalues of T are $1/2$ and 1; v_1 and v_3 are eigenvectors associated with $1/2$ and 1 respectively. The eigenspaces are $sp\,\{v_1\}$ and $sp\,\{v_3\}$. The generalized eigenspaces are

$$\begin{aligned}E_{1/2} &= sp\,\{v_1,\,v_2\}\\ E_1 &= sp\,\{v_3,\,v_4\}\ .\end{aligned}$$

(b) Show that R_4 is the direct sum of the generalized eigenspaces of T.

Since the vectors v_i span R_4 we have

$$\begin{aligned}R_4 &= \{c_1v_1+c_2v_2+c_3v_3+c_4v_4\}\\ &= \{c_1v_1+c_2v_2\}+\{c_3v_3+c_4v_4\}\\ &= E_{1/2}+E_1.\end{aligned}$$

So all we need to do now is to show that $E_{1/2}\cap E_1 = \left\{\vec{0}\right\}$. Suppose $x\in E_{1/2}\cap E_1$. This says there are constants c_i such that $x = c_1v_1+c_2v_2$ and $x = c_3v_3+c_4v_4$. Subtracting these two equations gives us

$$(c_1v_1+c_2v_2)-(c_3v_3+c_4v_4) = 0\ ,$$

but the v_i are linearly independent so $c_i = 0$ for each i. This tells us that $x = 0$.

(c) Find a formula for $T^n(\vec{v}_i)$ for each positive integer n.

$$\begin{aligned}T^nv_1 &= \frac{1}{2^n}v_1\\ T^nv_2 &= \frac{1}{2^n}v_2+\frac{n}{2^{n-1}}v_1\\ T^nv_3 &= v_3\\ T^nv_4 &= v_4+nv_3\end{aligned}$$

(d) Using the formulas you found in part a. determine when

$$\lim_{n\,\to\,\infty} T^n\left(c_1\vec{v}_1+c_2\vec{v}_2+c_3\vec{v}_3+c_4\vec{v}_4\right)$$

exists, and if the limit does exist what does it equal?

$$\begin{aligned}\lim_{n\,\to\,\infty} T^n\left(c_1\vec{v}_1+c_2\vec{v}_2+c_3\vec{v}_3+c_4\vec{v}_4\right) &= \lim_{n\,\to\,\infty}\left(c_1T^n\vec{v}_1+c_2T^n\vec{v}_2+c_3T^n\vec{v}_3+c_4T^n\vec{v}_4\right)\\ &= c_1\lim_{n\,\to\,\infty}\frac{1}{2^n}v_1+c_2\lim_{n\,\to\,\infty}\left(\frac{1}{2^n}v_2+\frac{n}{2^{n-1}}v_1\right)\\ &\quad +c_3\lim_{n\,\to\,\infty}v_3+c_4\lim_{n\,\to\,\infty}(v_4+nv_3)\end{aligned}$$

The only way a limit can exist for the last summand is if $c_4 = 0$, and in this case we have

$$\lim_{n\,\to\,\infty} T^n\left(c_1\vec{v}_1+c_2\vec{v}_2+c_3\vec{v}_3\right) = c_3v_3\ .$$

4. (30) The matrix A below is the matrix representation of $T : R^5 \to R^5$ with respect to the standard basis.

$$A = \begin{bmatrix} -1 & 1 & -4 & 3 & 1 \\ -4 & 2 & -9 & 6 & 3 \\ -5 & 2 & -12 & 8 & 3 \\ -5 & 2 & -13 & 9 & 3 \\ -5 & 3 & -13 & 9 & 3 \end{bmatrix}$$

(a) Find the characteristic polynomial of A.

$$p(\lambda) = \det(A - \lambda I) = -(\lambda+1)^2 (\lambda-1)^3 \ .$$

(b) What are the eigenvalues of A and their multiplicities ?

The eigenvalues of A are -1 and 1 with multiplicites 2 and 3 respectively.

(c) Find bases for each eigenspace and generalized eigenspace of A.

The matrices $(A - \lambda I)^j$ are row equivalent to the following matices

$$A + I \approx \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(A + I)^2 \approx \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(A + I)^3 \approx \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, the eigenspace for $\lambda = -1$, the null space of $A + I$ equals $sp\{\langle 0,0,1,1,1\rangle\}$. The generalized eigenspace for $\lambda = -1$ is the null space of $(A+I)^2 = E_{-1}$,

$$E_{-1} = sp\{\langle 0,1,1,1,0\rangle\ ,\ \langle 0,-1,0,0,1\rangle\} = sp\{v_1, v_2\}\,.$$

A similar analysis of the matrix $A - I$ and its powers leads to
Eigenspace of $\lambda = 1$ is $sp\{\langle 1,1,0,0,1\rangle\}$ and

$$\begin{aligned} E_1 &= sp\{\langle -2,1,1,0,0\rangle\,, \langle 1,-1,0,1,0\rangle\,, \langle 1,1,0,0,1\rangle\} \\ &= sp\{v_3, v_4, v_5\}\,. \end{aligned}$$

(d) From the bases found in c. construct a basis for R_5, and determine the matrix representation of T with respect to this basis.

Some computations lead to

$$\begin{aligned} Tv_1 &= -2v_1 - v_2 \\ Tv_2 &= v_1 \\ Tv_3 &= -v_4 \\ Tv_4 &= v_3 + 2v_4 + v_5 \\ Tv_5 &= v_5 \end{aligned}$$

Thus, the matrix representation of T with respect to this basis is

$$B = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Notice B has block diagonal form. If we call the 2×2 matrix in the upper left corner B_1 and B_2 the 3×3 matrix in the lower right corner it is an easy computation to see that

$$\begin{aligned} \det(B_1 - \lambda I) &= (\lambda + 1)^2 \\ \det(B_2 - \lambda I) &= -(\lambda - 1)^3 , \end{aligned}$$

which we expect from our results in problem 2.

5. (20) If V is a vector space over a field F, the dual of V, V^*, is the set of linear transformations from V into F.

$$V^* = \{T : V \to F, \text{ where } T \text{ is a linear transformation}\} .$$

If $\{\vec{v}_i\}_{i=1}^K$ is a basis of V, the dual basis of V^* is the set $\{\hat{v}_i\}_{i=1}^K$, where

$$\hat{v}_i(\vec{v}_j) = \delta_{ij} \text{ the Kronecker delta.}$$

(a) Show that V^* is a vector space over F.

The fact that V^* is a vector space follows from the fact that the set of linear transformation from a vector space V into a vector space W is a vector space. V^* is the particular case when W is F.

(b) Show that $\{\hat{v}_i\}_{i=1}^K$ is a basis of V^*, where $K = \dim(V)$. Thus, for finite dimensional vector spaces we have $\dim(V) = \dim(V^*)$.

The first thing to note is that if $\vec{x}$ is any vector in V, then $\vec{x} = \sum_{i=1}^k c_i \vec{v}_i$, for some constants c_i. Thus, for any j,

$$\hat{v}_j(\vec{x}) = \sum_{i=1}^k c_i \hat{v}_j(\vec{v}_i) = \sum_{i=1}^k c_i \delta_{i,j} = c_j .$$

That is, $\hat{v}_j$ is that linear transformation, which picks out the j^{th} coordinate of $\vec{x}$ with respect to the basis $\{\vec{v}_i\}_{i=1}^k$. To see that the $\hat{v}_j$'s form a basis of V^* we need to see that they are linearly independent and spanning. So suppose λ_j are such that $\hat{0} = \sum_{i=1}^{k} \lambda_i \hat{v}_i$, where $\hat{0}$ represents the zero linear transformation. Then for each j we have

$$\vec{0} = \hat{0}(\vec{v}_j) = \sum_{i=1}^{k} \lambda_i \hat{v}_i(\vec{v}_j) = \sum_{i=1}^{k} \lambda_i \delta_{i,j} = \lambda_j.$$

Thus, the $\hat{v}_j$'s are linearly independent. If T is any linear transformation in V^*, we want to show that T can be written as a linear combination of the $\hat{v}_j$'s. Set $\lambda_j = T(\vec{v}_j)$, and $\hat{T} = \sum_{j=1}^{k} \lambda_j \hat{v}_j$. Then both T and $\hat{T}$ agree each basis vector $\vec{v}_i$ for $i = 1, \cdots, k$. Thus, $T = \hat{T}$, and the set $\{\hat{v}_i\}_{i=1}^{k}$ spans V^*.

6. (30)Let T_1 be the linear transformation of R_3 that reflects R_3 through the x-y plane and let T_2 be the linear transformation of R_3 that rotates R_3 45° in a counter clockwise direction about the line through the origin and the point $\langle 1,\ 2,\ 1\rangle$, with counterclockwise being determined by standing at the point $\langle 1,\ 2,\ 1\rangle$ and looking towards the origin.

   (a) Find the matrix representations of T_1 and T_2 with respect to the standard basis of R_3.

The determination of T_1 is easy. Let $\vec{e}_i$ denote, as usual, the standard basis of R_3. Then we have

$$\begin{aligned} T(\vec{e}_1) &= \vec{e}_1 \\ T(\vec{e}_2) &= \vec{e}_2 \\ T(\vec{e}_3) &= -\,\vec{e}_3 \end{aligned}$$

Letting A_i denote the matrix representations of T_i with respect to the standard basis, the above equations tell us that

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

To determine A_2, we first find the matrix representation of T_2 with respect to a basis that makes sense for this transformation. Set $\vec{v}_1 = \langle 1,\ 2,\ 1\rangle$. We next want to pick $\vec{v}_2$ and $\vec{v}_3$ that are a basis for the subspace, i.,e., plane, perpendicular to $\vec{v}_3$. The equation for this plane is $x + 2y + z = 0$. Set $\vec{x}_2 = \langle 1,\ 0,\ -1\rangle$, and $\vec{x}_3 = \langle -2,\ 2,\ -2\rangle$. Unfortunately, while these vectors are perpendicular to each other and you can obtain $\vec{x}_3$ by rotating $\vec{x}_2$ counterclock wise 90°, they don't have the same length, which has to be true if we're going to use the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ in building A_2. So divide each of the $\vec{x}_i$'s by their length to get the following vectors

$$\begin{aligned} \vec{v}_1 &= \langle 1,\ 2,\ 1\rangle \\ \vec{v}_2 &= \left\langle \frac{1}{\sqrt{2}},\ 0,\ \frac{-1}{\sqrt{2}} \right\rangle \\ \vec{v}_3 &= \left\langle \frac{-1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{-1}{\sqrt{3}} \right\rangle \end{aligned}$$

Then we have $T_2(\vec{v}_1) = \vec{v}_1$, $T_2(\vec{v}_2) = \cos(45)\,\vec{v}_2 + \sin(45)\,\vec{v}_3$, and $T_2(\vec{v}_3) = -\sin(45)\,\vec{v}_2 + \cos(45)\,\vec{v}_3$. Thus, the matrix representation of T_2 with respect to this basis is

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

Setting P equal to the change of basis matrix for which $[x]_S = P\,[x]_V$, we have

$$A_2 = PBP^{-1} = \begin{bmatrix} \frac{2+5\sqrt{2}}{12} & \cdots & \cdots \\ \frac{2+\sqrt{3}-\sqrt{2}}{6} & \cdots & \cdots \\ \frac{2-\sqrt{2}-4\sqrt{3}}{12} & \cdots & \cdots \end{bmatrix},$$

where the last two columns of A_2 have been left blank.

(b) Does it make a difference if rotate and then reflect or if reflection is followed by rotation? That is, are T_1T_2 and T_2T_1 equal or different?

A quick check of T_1T_2 and T_2T_1 on $\vec{e}_1$ shows that the two transformations do not commute.

$$T_2T_1(\vec{e}_1) = T_2(\vec{e}_1) = \frac{2+5\sqrt{2}}{12}\vec{e}_1 + \frac{2+\sqrt{3}-\sqrt{2}}{6}\vec{e}_2 + \frac{2-\sqrt{2}-4\sqrt{3}}{12}\vec{e}_3$$

while

$$\begin{aligned} T_1T_2(\vec{e}_1) &= T_1\left(\frac{2+5\sqrt{2}}{12}\vec{e}_1 + \frac{2+\sqrt{3}-\sqrt{2}}{6}\vec{e}_2 + \frac{2-\sqrt{2}-4\sqrt{3}}{12}\vec{e}_3\right) \\ &= \frac{2+5\sqrt{2}}{12}\vec{e}_1 + \frac{2+\sqrt{3}-\sqrt{2}}{6}\vec{e}_2 - \frac{2-\sqrt{2}-4\sqrt{3}}{12}\vec{e}_3\ . \end{aligned}$$

Since $\frac{2-\sqrt{2}-4\sqrt{3}}{12} \neq -\frac{2-\sqrt{2}-4\sqrt{3}}{12}$, the linear transformations do not commute.