1. (15) Define the following:

a. the span of the set of vectors $\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_k\}$,

The span of a set of vectors is the set of all possible linear combinations of the vectors. That is,

$$\text{Span}\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_k\} = \left\{ \sum_{i=1}^{k} \alpha_i \vec{x}_i \;:\; \alpha_i \in K \right\},$$

where K denotes the set of scalars.

b. linear transformation

A linear transformation from one vector space V to another W is a function $L : V \to W$ such that

$$L(\vec{x} + \vec{y}) = L(\vec{x}) + L(\vec{y}) \text{ for every } \vec{x}, \vec{y} \text{ in } V$$
$$L(\alpha\vec{x}) = \alpha L(\vec{x}) \text{ for all } \alpha \in K \text{ and } \vec{x} \in V.$$

c. subspace.

T is a subspace of a vector space V if T is a subset of V and, using the same operations of vector addition and scalar multiplication as in V, is a vector space also.

2. (15) Suppose $S = \{\vec{x}_1, \cdots, \vec{x}_k\}$ is a linearly independent subset of the vector space V. Let L be a linear transformation from V to W, which is one-to-one. Show that the set $\{L(\vec{x}_1), \cdots, L(\vec{x}_k)\}$, is a linearly independent subset of the vector space W. Remember: one-to-one means $L(\vec{x}) = L(\vec{y})$ can only happen if $\vec{x} = \vec{y}$.

Suppose there are constants c_i, $1 \le i \le k$ such that

$$\sum_{i=1}^{k} c_i L(\vec{x}_i) = \vec{0}.$$

Since L is linear, this equation implies

$$\vec{0} = \sum_{i=1}^{k} c_i L(\vec{x}_i) = L\left[\sum_{i=1}^{k} c_i \vec{x}_i\right].$$

Since $L(\vec{0}) = \vec{0}$ and L is one-to-one, this says that $\sum_{i=1}^{k} c_i \vec{x}_i = \vec{0}$. Since the $\vec{x}_i$ are linearly independent each $c_i = 0$. Thus, the set $\{L(\vec{x}_1), \cdots, L(\vec{x}_k)\}$ is also linearly independent.

3. (30)Let V be the vector space of all 2×2 real valued matrices. Let B be the set defined below, and let S be the standard basis of $V = M_{2,2}$.

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -3 \\ 1 & 7 \end{bmatrix} \right\}$$

a. Show that the set B is a basis of V.

Call the elements of the set B, $\vec{b}_i$ for $i = 1$ to 4. Then $\vec{b}_2$ and $\vec{b}_3$ can be used to make $\vec{s}_1$ the first vector in the standard basis S. Using $\vec{s}_1$ and $\vec{b}_1$ we can make $\vec{s}_3$. Then $\vec{s}_3$ and $\vec{b}_3$ we can make $\vec{s}_2$, and $\vec{s}_2$ with $\vec{s}_3$ and $\vec{b}_4$ make $\vec{s}_4$. Thus, the span of the set B contains the standard basis S, and this means the span is all of V. Since $\dim(V) = 4$ and B is a spanning set with 4 vectors, it must be a basis of V.

b. Find the change of basis matrix P such that for $\vec{x} \in V$ we have

$$[\vec{x}]_S = P[\vec{x}]_B ,$$

where $[\vec{x}]_S$ denotes the coordinates with respect to the basis S, etc.

$$P = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & -3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix}.$$

Note that the columns of P are the coordinates of the vectors in B with respect to the standard basis.

c. Show how to use the matrix P to find the coordinates of the matrix $\begin{bmatrix} 11 & 29 \\ -37 & 12 \end{bmatrix}$ with respect to the basis B.

From the equation which defines P we have

$$[\vec{x}]_B = P^{-1}[\vec{x}]_S = P^{-1}\begin{bmatrix} 11 & 29 & -37 & 12 \end{bmatrix}^T$$

d. If $\hat{p}_{i,j}$ denotes the entry in the i,j position of the matrix P^{-1}, what is the value of $\hat{p}_{1,2}$? Hint, do not compute the matrix P^{-1}.

Use the fact that the inverse of a matrix can be found by computing the adjoint of the matrix. The i-j entry of a matrix is proportional to $\det(M_{j,i})$, where $M_{j,i}$ is the matrix obtained by deleting the j^{th} row and i^{th} column of the matrix. Thus,

$$\hat{p}_{1,2} = (-1^3)\frac{\det(M_{2,1})}{\det(P)}$$

$$= -\frac{\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 1 \\ 0 & 0 & 7 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & -3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix}\right)}$$

$$= -\frac{-7}{-35} = \frac{-1}{5} .$$

4. (20) Let $L : V \rightarrow W$ be a linear transformation between the two vector spaces V and W.

a. Define the kernel of L, and show that it is a subspace of V.

The kernel of L is the set of vectors annihilated by L. That is,

$$\ker(L) = \left\{\vec{x} \in V : L(\vec{x}) = \vec{0}\right\} .$$

b. If $\{\vec{x}_i\}_{i=1}^{n}$ is a basis of V, show that $\{L(\vec{x}_i)\}_{i=1}^{n}$ is a spanning set of the range of L.

A vector $\vec{y}$ is in the range of L if there is a vector $\vec{x} \in V$ such that $L(\vec{x}) = \vec{y}$. So suppose $\vec{y} \in \mathrm{Rg}(L)$. Then

$$\vec{y} = L(\vec{x}) = L\left(\sum_{i=1}^{n} c_i\vec{x}_i\right)$$

$$= \sum_{i=1}^{n} c_i L(\vec{x}_i) .$$

Thus, any vector $\vec{y}$ in the range of L is a linear combination of the vectors in the set $\{L(\vec{x}_i)\}_{i=1}^{n}$. That is, this set spans the range of L.

c. What else can you say about $\{L(\vec{x}_i)\}_{i=1}^{n}$ if the kernel of L is just the zero vector?

If the kernel of L is just the zero vector than this set is linearly independent. See proof of problem 2.

5. (20) Let $V = M_{2,3}$ and $W = P_5$. Suppose A is the matrix representation of a linear transformation $L : V \to W$ with respect to the bases

$$\left\{ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 1 \\ 3 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \right\}$$

$$\left\{ t, 2 + t^2, 3 - t, t^4 + 1, t^3 - t^4 \right\}$$

of V and W respectively. Assume

$$A = \begin{bmatrix} 1 & -1 & 0 & 2 & 3 & 1 \\ 2 & 0 & 3 & -2 & 2 & -3 \\ 3 & 3 & 1 & 2 & 1 & 0 \\ 4 & 5 & 2 & -2 & 0 & 1 \\ -5 & 7 & -3 & 2 & -12 & 4 \end{bmatrix}.$$

Compute $L\left(\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)$.

The first thing is to note that the coordinates of $\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ with respect to the given basis of V are

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \end{bmatrix}^T.$$

Thus, the coordinates of $L\left(\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)$ with respect to the given basis of P_5 are

$$A \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 2 & 3 & 1 \\ 2 & 0 & 3 & -2 & 2 & -3 \\ 3 & 3 & 1 & 2 & 1 & 0 \\ 4 & 5 & 2 & -2 & 0 & 1 \\ -5 & 7 & -3 & 2 & -12 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -2 & 0 & 1 & 12 \end{bmatrix}^T.$$

Hence,

$$L\left(\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = (-2)t + (-2)(2 + t^2) + (0)(3 - t) + (1)(t^4 + 1) + (12)(t^3 - t^4)$$

$$= -11t^4 + 12t^3 - 2t^2 - 2t - 3 .$$