1. (15) Define the following:

(a) the set of vectors $\{\vec{x}_1, \cdots, \vec{x}_k\}$ spans a vector space V,

To say that $S = \{\vec{x}_1, \cdots, \vec{x}_k\}$ spans V means that every vector in V can be written as a linear combination of the vectors in S.

(b) null space of an $m \times n$ matrix A,

The null space of A is the set of vectors $\vec{x} \in R^n$ such that $A\vec{x} = \vec{0}$.

(c) the set of vectors $\{\vec{x}_1, \cdots, \vec{x}_k\}$ is a basis of the vector space V.

A basis of V is a linearly independent set that also spans V.

2. (25) Consider the following set of equations:

$$\begin{aligned} 2x_1 - x_3 + x_4 &= 0 \\ x_1 + x_2 + x_3 + x_4 &= 1 \\ x_1 - x_2 - x_3 - x_4 &= 3 \end{aligned}$$

(a) Express the solution set of this system of equations in terms of the fewest number of free variables possible.

The augmented matrix of this system is

$$A = \begin{bmatrix} 2 & 0 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 3 \end{bmatrix}.$$

It is row equivalent to the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 & -5 \\ 0 & 0 & 1 & -1 & 4 \end{bmatrix}.$$

Thus, we have $x_1 = 2$, $x_2 = -5 - 2x_4$, and $x_3 = 4 + x_4$. So the solution set is

$$\begin{aligned} S &= \{(2, \ -5 - 2x_4, \ 4 + x_4, \ x_4) : x_4 \in R\} \\ &= \{(2, \ -5, \ 4, \ 0) + x_4 \, (0, \ -2, \ 1, \ 1)\}. \end{aligned}$$

The solution set is expressed in terms of the single free variable x_4.

(b) Is the solution set a subspace of R^4?

The solution set is not a subspace. It is non-empty, but since the $\vec{0}$ vector is not in S, S cannot be a subspace.

3. (25) Let $A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 2 & 5 \\ -5 & 5 & 5 \end{bmatrix}$.

(a) Find a basis for the null space of A.

The matrix A is row equivalent to

$$A \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, the null space of A is the set

$$\{(-x_3,\ -2x_3,\ x_3)\} = \text{ span}\{(-1,\ -2,\ 1)\}.$$

So, the set consisting of the single vector $(-1,\ -2,\ 1)$ is a basis for the null space of A.

(b) What is the dimension of the null space of A?

Since the null space has a basis with one vector the dimension of the null space is 1.

4. (25) Let $S = \left\{1 - t,\ t^2,\ 2 + 3t\right\} \subseteq P_4$.

(a) Does S span P_4?

S does not span P_4. P_4 is the set of all polynomials of degree 3 or less. No linear combination of the vectors in S can equal the polynomial t^3, which is in P_4.

(b) Is S linearly independent?

S is linearly independent. For suppose there are constants c_1, c_2, and c_3 such that

$$\begin{aligned} c_1(1-t) + c_2\left(t^2\right) + c_3(2+3t) &= \vec{0} \\ (c_1 + 2c_3) + (-c_1 + 3c_3)\,t + c_2 t^2 &= \vec{0} \end{aligned}$$

This last equation is valid if and only if

$$\begin{aligned} c_1 + 2c_3 &= 0 \\ -c_1 + 3c_3 &= 0 \\ c_2 &= 0 \end{aligned}$$

The only solution to this system is the trivial one: $c_1 = c_2 = c_3 = 0$.

(c) Find a basis of P_4 that contains the set S.

The set S is linearly independent and its span does not contain t^3. Thus, the set

$$\left\{1 - t,\ t^2,\ 2 + 3t,\ t^3\right\}$$

is linearly independent. Since this set contains 4 vectors and the dimension of P_4 is 4, it must be a basis.

5. (10) Let $A = \begin{bmatrix} 2 & 0 & 0 & 3 \\ 3 & 1 & 1 & -2 \\ 5 & 0 & 1 & 0 \\ 0 & -3 & 0 & -5 \end{bmatrix}$.

(a) Calculate the value of the determinant of A. For your first step use expansion by cofactors with the third column.

$$\begin{aligned}
\det(A) &= -\det\left(\begin{bmatrix} 2 & 0 & 3 \\ 5 & 0 & 0 \\ 0 & -3 & -5 \end{bmatrix}\right) + \det\left(\begin{bmatrix} 2 & 0 & 3 \\ 3 & 1 & -2 \\ 0 & -3 & -5 \end{bmatrix}\right) \\
&= -(-5)\det\left(\begin{bmatrix} 0 & 3 \\ -3 & -5 \end{bmatrix}\right) + 2\det\left(\begin{bmatrix} 1 & -2 \\ -3 & -5 \end{bmatrix}\right) - 3\det\left(\begin{bmatrix} 0 & 3 \\ -3 & -5 \end{bmatrix}\right) \\
&= 45 + 2(-5-6) - 3(9) \\
&= -4.0
\end{aligned}$$

(b) What is the entry in the 1, 2 position of A^{-1}.
The inverse of a matrix is given by the formula

$$A^{-1} = \frac{1}{\det A}\left[(-1)^{i+j}\det M_{ij}\right]^T$$

where M_{ij} is the matrix obtained by deleting the i^{th} row and j^{th} column of A. Thus, the entry in the 1,2 position of A^{-1} will be

$$\begin{aligned}
\frac{1}{\det A}(-1)^{2+1}\det M_{2,1} &= \frac{1}{-4}\det\begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 0 \\ -3 & 0 & -5 \end{bmatrix} \\
&= \frac{1}{4}(3)(3) \\
&= \frac{9}{4}
\end{aligned}$$