**1.** (15) Define the following:

**a.** a linear transformation

Let $T$ be a mapping from the vector space $V$ into the vector space $W$. $T$ is a linear transformation if for every $\vec{x}$ and $\vec{y} \in V$ and $\alpha \in R$ we have

$$T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$$
$$T(\alpha\vec{x}) = \alpha T(\vec{x}) .$$

**b.** matrix representation of a linear transformation

If $T$ is a linear transformation from $V$ into $W$ and $B_1$ and $B_2$ are bases of $V$ and $W$ respectively, then $A$ is a matrix representation of $T$ with respect to the two bases if

$$[T\vec{x}]_{B_2} = A[\vec{x}]_{B_1} .$$

**c.** row space of a matrix

Let $\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_m$ be the $m$ rows of an $m \times n$ matrix. That is, each of the $\vec{r}_i$ can be thought of as a vector in $R^n$. The row space of $A$ is the span of these row vectors.

**2.** (20) $S = \{1, t, t^2\}$ is the standard basis of $P_3$, the vector space of polynomials of degree 2 or less. Let $U = \{t^2 + 2t, 6 - t^2, 1 + t + t^2\}$.

**a.** Show that $U$ is a basis of $P_3$.

Since the dimension of $P_3$ is 3, and the set $U$ contains 3 vectors, it will suffice to show that $U$ is either linearly independent or spanning. To see that this set is linearly independent suppose there are constants $c_i$ such that

$$c_1(t^2 + 2t) + c_2(6 - t^2) + c_3(1 + t + t^2) = \vec{0}$$
$$(6c_2 + c_3) + (2c_1 + c_3)t + (c_1 - c_2 + c_3)t^2 = \vec{0} .$$

This polynomial can be the zero polynomial if and only if

$$6c_2 + c_3 = 0$$
$$2c_1 + c_3 = 0$$
$$c_1 - c_2 + c_3 = 0 .$$

These equations have only one solution: $c_1 = 0$, $c_2 = 0$, $c_3 = 0$. Thus, the polynomials in the set $U$ are linearly independent.

**b.** If $[\vec{x}]_U = [3, -2, 5]$, what is $\vec{x}$?

$$\begin{aligned}\vec{x} &= 3(t^2 + 2t) - 2(6 - t^2) + 5(1 + t + t^2) \\ &= -7 + 11t + 10t^2\ ,\end{aligned}$$

**c.** Find the change of basis matrix $P$ such that $[\vec{x}]_S = P[\vec{x}]_U$.

$$P = \begin{bmatrix} 0 & 6 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

**d.** Using the matrix $P$, calculate $[1 + 2t + 3t^2]_U$.

From $[\vec{x}]_S = P[\vec{x}]_U$, we have $[\vec{x}]_U = P^{-1}[\vec{x}]_s$. So

$$\begin{aligned}[1 + 2t + 3t^2]_U &= \begin{bmatrix} 0 & 6 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{8} & \frac{7}{8} & -\frac{3}{4} \\ \frac{1}{8} & \frac{1}{8} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{3}{4} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{5}{8} \\ -\frac{3}{8} \\ \frac{13}{4} \end{bmatrix}.\end{aligned}$$

**3.** (15) Various mappings are defined below, determine which, if any, are linear transformations.

**a.** $T : R^2 \to R^1,\ T(x_1, x_2) = x_1$.

This is a linear transformation, as is easily checked.

**b.** $T : R^2 \to R^1,\ T(x_1, x_2) = x_1 x_2$.

This is not a linear transformation. As

$$T(2(x_1, x_2)) = T((2x_1, 2x_2)) = 4x_1x_2 \neq 2T((x_1, x_2))\ .$$

**c.** $T : R^2 \to M_{2,3},\ T(x_1, x_2) = \begin{bmatrix} x_1 & 0 & 1 \\ 0 & x_2 & 0 \end{bmatrix}$.

This too is not a linear transformation. A quick check shows that $T(2\vec{x}) \neq 2T(\vec{x})$.

**4.** (25) Let $T$ denote the linear transformation of $R^3$ into itself that first reflects $R^3$ through the plane $x_1 + x_2 - 2x_3 = 0$, and then rotates $R^3$ about the $z$-axis $60^\circ$ in a counter clockwise direction when viewed from the positive $z$-axis towards the origin. Hint: if $T = T_2T_1$, then the matrix representation of $T$ is $A_2A_1$ where $A_i$ is the matrix representation of $T_i$.

**a.** Find the matrix representation of $T$ with respect to the standard basis.

Let's find the matrix representation of $T_1$, the transformation that reflects through the plane, with respect to the standard basis first. To do this easily we first find a matrix representation of $T_1$ with respect to the basis $B_1 = \{(0, 2, 1), (2, 0, 1), (1, 1, -2)\}$ If $\hat{A}_1$ denotes this matrix, then

$$\hat{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

If $P$ denotes the change of basis matrix $[\vec{x}]_S = P[\vec{x}]_{B_1}$, then

$P = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & -2 \end{bmatrix}$, and the matrix we are looking for $A_1$, which represents $T_1$ with respect to the standard basis is

$$\begin{aligned} A_1 &= P\hat{A}_1P^{-1} \\ &= \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & -2 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} -\frac{1}{12} & \frac{5}{12} & \frac{1}{6} \\ \frac{5}{12} & -\frac{1}{12} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}. \end{aligned}$$

The matrix $A_2$ equals

$$A_2 = \begin{bmatrix} \cos 60 & -\sin 60 & 0 \\ \sin 60 & \cos 60 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We're now able to compute the matrix representation $A$ of $T$ with respect to the standard basis.

$$\begin{aligned} A &= A_2 A_1 \\ &= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} & 0 \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{6}\sqrt{3} + \frac{1}{3} & -\frac{1}{3}\sqrt{3} - \frac{1}{6} & -\frac{1}{3}\sqrt{3} + \frac{1}{3} \\ \frac{1}{3}\sqrt{3} - \frac{1}{6} & -\frac{1}{6}\sqrt{3} + \frac{1}{3} & \frac{1}{3}\sqrt{3} + \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}. \end{aligned}$$

**b.** Do we get the same linear transformation if the rotation is performed first?

The answer is no. If the rotation is performed first, then the matrix representation for this compostion of the two transformations is $A_1 A_2$, and this product equals

$$A_1 A_2 = \begin{bmatrix} -\frac{1}{6}\sqrt{3} + \frac{1}{3} & -\frac{1}{3}\sqrt{3} - \frac{1}{6} & \frac{2}{3} \\ \frac{1}{3}\sqrt{3} - \frac{1}{6} & \frac{1}{6}\sqrt{3} + \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3}\sqrt{3} + \frac{1}{3} & -\frac{1}{3}\sqrt{3} + \frac{1}{3} & -\frac{1}{3} \end{bmatrix},$$

and this does not equal the matrix $A_2 A_1$.

**5.** (25) Let $T$ denote a linear transformation of $P_3 \to P_2$. Let $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 0 & 5 \end{bmatrix}$ be the matrix representation of $T$ with respect to the standard basis $S = \{1, t, t^2\}$ in $P_3$ and the basis $B = \{2t, 3 - t\}$ in $P_2$.

**a.** $T(6 - t + t^2)$ equals?

$$[T(6 - t + t^2)]_B = A[6 - t + t^2]_S$$

$$= \begin{bmatrix} 2 & 1 & -1 \\ 3 & 0 & 5 \end{bmatrix} \begin{bmatrix} 6 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ 23 \end{bmatrix}.$$

Thus, $T(6 - t + t^2) = 10(2t) + 23(3 - t) = 69 - 3t$.

**b.** Find a basis for the kernel of $T$.

The matrix $A$ is row equivalent to the matrix $\begin{bmatrix} 1 & 0 & \frac{5}{3} \\ 0 & 1 & -\frac{13}{3} \end{bmatrix}$. The rank of this matrix is 2, hence its null space has dimension 1 and thus, the kernel of $T$ also has dimension 1. Since the vector $(-5, 13, 3)$ spans the null space of $A$, we know that the vector

$$-5 + 13t + 3t^2$$

will span the kernel of $T$.

**c.** Find a basis for the image of $T$.

A basis for the column space of $A$ is $\{(2, 3), (1, 0)\}$. Thus a basis for the image of $T$ is

$$\{2(2t) + 3(3 - t), 2t\} = \{t + 9, 2t\}.$$