Math 304

Key to Exam 1

(15) 1. Consider the system of equations:

$$\begin{aligned} 2x_1 - x_2 + x_3 - x_4 &= 5 \\ 6x_1 + x_3 + 2x_4 &= 1 \end{aligned}$$

Find the solution set of this system.

The augmented matrix of this system is row equivalent to the matrix: $\begin{bmatrix} 1 & 0 & 1/6 & 1/3 & 1/6 \\ 0 & 1 & -2/3 & 5/3 & -14/3 \end{bmatrix}$. This says that the solution set is the subset (not subspace) of R^4 which satisfies these equations. That is,

$$\text{Solution set } = \{\, (-x_3/6 - x_4/3 + 1/6, 2x_3/3 - 5x_4/3 - 14/3, x_3, x_4) \;:\; x_3,\, x_4 \in R \,\}\,.$$

Thus, an arbitrary solution has the form $x_3(-1/6, 2/3, 1, 0) + x_4(-1/3, -5/3, 0, 1) + (1/6, -14/3, 0, 0)$.

(15) 2. Let $A = \begin{bmatrix} 8 & 10 & 2 \\ 12 & 14 & 1 \\ 8 & 11 & 16 \end{bmatrix}$. Find an LU factorization of the matrix A in which the lower triangular matrix L has only ones on its main diagonal. Explain how you got this factorization.

The first thing to be done is to zero out the entries in the first column below the pivot point. Thus, let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3/2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$. Then $E_1A = \begin{bmatrix} 8 & 10 & 2 \\ 0 & -1 & -2 \\ 0 & 1 & 14 \end{bmatrix}$. Now let $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Then $E_2E_1A = \begin{bmatrix} 8 & 10 & 2 \\ 0 & -1 & -2 \\ 0 & 0 & 12 \end{bmatrix}$. Call this last matrix U and note that it is upper triangular. Then we have $A = E_1^{-1}E_2^{-1}U$. The matrix $L = E_1^{-1}E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$, is lower triangular and we have $A = LU$.

(20) 3. let $W = \left\{ A \in M_{2,2} \;:\; A \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$.

a. Show that W is a subspace of $M_{2,2}$.

To see that W is a subspace we first note that the zero 2×2 matrix is in W. Thus, W is not empty. Now suppose that the matrices A and B are both in W. We need to show that their sum is in W. $(A + B)\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} = A\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} +$

$B\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Thus, the sum of two arbitrary vectors in W is back in W. We now show that W is closed under scalar multiplication. Thus, let α be an arbitrary scalar and A any vector in W. Then, $(\alpha A)\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} = (\alpha)\left(A\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}\right) = (\alpha)\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Thus, W is closed under scalar multiplication and we may now conclude that W is indeed a subspace of $M_{2,2}$.

b. Find a basis of W.

From the equations describing W, we see that a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W$ if and only if $a = b$, and $c = d$. Thus, a basis for W is the following set which consists of two matrices:

$$\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$

(30) 4. Let $B = \{ (1,2,3),\ (2,0,1),\ (1,1,1) \}$.

a. Show that B is a basis of R^3.

It is an easy matter to see that these three vectors are linearly independent. Place them as columns in a matrix. The reduced row echelon form of that matrix is the 3×3 identity matrix. This means that the only solution to the equations

$$c_1(1,2,3) + c_2(2,0,1) + c_3(1,1,1) = (0,0,0)$$

is $c_1 = 0,\ c_2 = 0,$ and $c_3 = 0$.

b. If the coordinates of $\vec{x}$ with respect to the basis B are $[1,2,3]$, what vector is $\vec{x}$?

$\vec{x} = 1(1,2,3) + 2(2,0,1) + 3(1,1,1) = (8,5,8)$.

c. Find the coordinates of $(3,2,1)$ with respect to the basis B.

Solving the system $c_1(1,2,3) + c_2(2,0,1) + c_3(1,1,1) = (3,2,1)$, we get that $c_1 = -1,\ c_2 = 0,$ and $c_3 = 4$. Thus, $[(3,2,1)]_B = [-1,0,4]$.

d. Let P be the change of basis matrix which converts coordinates with respect to the basis B into coordinates with respect to the standard basis of R^3. That is, $[\vec{x}]_S = P[\vec{x}]_B$. Find the matrix P, and use it to check your answer for part c.

The change of basis matrix $P = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 3 & 1 & 1 \end{bmatrix}$. Checking the answer from part c. we have $P^{-1}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$.

(15) 5. There are three towns: Bryan, College Station, and Navasota with a combined population of 99,000 residents. This number remains constant from year to year even though the individual populations of each of the three cities vary. Suppose that each year 20% of the residents of Bryan move to Navasota and 10% move to College Station, while the remaining residents of Bryan stay in Bryan. Suppose also that each year 10% of the College Station residents move to Bryan, 5% move to Navasota, and the rest stay in College Station. We also have that each year 20% of the residents of Navasota move to College Station, another 20% move to Bryan, and the rest stay in Navasota. Suppose that the initial population distribution of these three cites is: Bryan: 45,000; College Station: 35,000; and Navasota: 19,000.

a. If A is the matrix which tells you how to go from the population distribution in one year to the distribution in the next year, what is A? That is, if $\vec{x}_k$ is the population distribution at the end of year k, then we have $\vec{x}_{k+1} = A\vec{x}_k$.

Let b_k, c_k, and n_k denote the populations of Bryan, College Station, and Navasota at the end of the k^{th} year. Then the equations which relate the population distribution from year $k-1$ to year k are:

$$\begin{aligned} b_k &= .7b_{k-1} + .1c_{k-1} + .2n_{k-1} \\ c_k &= .1b_{k-1} + .85c_{k-1} + .2n_{k-1} \\ n_k &= .2b_{k-1} + .05c_{k-1} + ..6n_{k-1} \end{aligned}$$

Thus, the matrix we want is $A = \begin{bmatrix} .7 & .1 & .2 \\ .1 & .85 & .2 \\ .2 & .05 & .6 \end{bmatrix}$.

b. Ten years from the initial year, what will the population of each of the three cities be?

The easiest way to determine the population distribution 10 years from the initial distribution is to compute A^{10} and then multiply this matrix by the initial distribution. When you do this, the result is:

$$b_{10} \approx 30,234, \ c_{10} \approx 47,538, \ n_{10} \approx 21,228$$

(5) 6. Suppose the set of vectors $S = \{\vec{x}_1, \vec{x}_2\}$ is linearly independent. Suppose that the vector $\vec{y}$ is not in the span of the set S. Show that the set $\{\vec{x}_1, \vec{x}_2, \vec{y}\}$ is also linearly independent.

Suppose there are constants $c_1, c_2,$ and c_3 such that $c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{y} = \vec{0}$. There are two cases to consider.

1. Suppose that the coefficient of $\vec{y}$ is not zero. Then we can solve the above equation for $\vec{y}$ as a linear combination of the vectors $\vec{x}_1$ and $\vec{x}_2$. This contradicts the fact that $\vec{y}$ is not in the span of the given two vectors.

2. From the above argument we see that c_3 is zero. This then says that we have a linear combination of the vectors $\vec{x}_1$ and $\vec{x}_2$ which equals zero. Since these two vectors are linearly independent, we must have $c_1 = 0$, and $c_2 = 0$.

Thus, the set consisting of $\vec{x}_1$, $\vec{x}_2$, and $\vec{y}$ is linearly independent.