Math 304
Final Exam Key
Dec. 8, 1995

(20) 1. Let $V = P_3$ the vector space of polynomials of degree 2 or less. Let $B_1 = \{1+t,\ 1+t^2,\ t+t^2\}$. Let $B_2 = \{1+t+2t^2,\ -2+3t+t^2,\ 6-t\}$.

a. What is the definition of a basis?

A basis of a vector space is a set of vectors that is both spanning and linearly independent.

b. Show that the set B_1 is a basis of the vector space V?

It is an easy matter to verify that the vectors in B_1 are linearly independent. See below. To verify that this set also spans P_3 we note that B_1 contains three vectors and that the dimension of P_3 is three. Thus, once we know that B_1 is linearly independent, we know that it must also span. Suppose now that there are constants c_i such that $c_1(1+t) + c_2(1+t^2) + c_3(t+t^2) = \vec{0}$. This leads to the equations:

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 + c_3 &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

One quickly sees that the only solution to this system is the trivial one. That is, all of the c_i's are zero.

c. Let P be the change of basis matrix such that $[\vec{x}]_{B_1} = P[\vec{x}]_{B_2}$. What does the first column of P equal? You do not have to calculate the first column, just tell me what it must equal.

The first column of P must consist of the coordinates with respect to the basis B_1 of the first vector in the basis B_2. Thus, the first column is $[0, 1, 1]^T$.

d. Define $L : P_3 \to M_{2,2}$ by $L(\vec{p}) = \begin{bmatrix} p(1) & p(1)+1 \\ p(2) & p(3) \end{bmatrix}$. Is L a linear transformation?

The formula which describes the entry in the 1-2 position of the matrix is not a linear function. Thus, L is not a linear transfromation. One can also check that $L(p_1 + p_2) \neq L(p_1) + L(p_2)$.

(20) 2. Let A be an $m \times n$ matrix of real numbers. Let $\vec{b} \in R^m$ and $\vec{x} \in R^n$

a. Define the nullspace of the matrix A.

The null space of a matrix is the set of $\vec{x}$'s such that $A\vec{x} = \vec{0}$.

b. Define the column space , $CS(A)$, of the matrix A.

The column space of a matrix is the span of the columns of the matrix.

c. Explain why the equation $A\vec{x} = \vec{b}$ has a solution if and only if $\vec{b} \in CS(A)$.

Supppose $\vec{b}$ is in the column space of A, then there are constants x_i such that $x_1\vec{C}_1 + \cdots + x_n\vec{C}_n = \vec{b}$, where $\vec{C}_i$ is the i^{th} column of the matrix A. This is the same as saying that $A\vec{x} = \vec{b}$, where $\vec{x}$ is that vector whose entries are the above x_i. Conversely if $A\vec{x} = \vec{b}$, then the entries in the vector $\vec{x}$ are scalars which can be used to express $\vec{b}$ as a linear combination of the columns of A.

d. If $\vec{x}_0$ is a solution to the system $A\vec{x} = \vec{b}$ and $\vec{\eta}$ is in th null space of A, show that $A\left(\vec{x}_0 + \vec{\eta}\right) = \vec{b}$.

$$A\left(\vec{x}_0 + \vec{\eta}\right) = A\vec{x}_0 + A\vec{\eta} = \vec{b} + \vec{0} = \vec{b}.$$

(20) 3. Let $L : P_3 \to M_{2,2}$ be the linear transformation such that $L(\vec{p}) = \begin{bmatrix} p(1) & 2p(1) \\ p(-1) & 2p(-1) \end{bmatrix}$.

Let $B_1 = \{\, 1+t,\ 1+t^2,\ t+t^2 \,\}$.

Let $B_2 = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

a. Find the matrix representation of L with respect to the bases B_1 and B_2.

$$\begin{aligned} L(1+t) &= \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}, \quad L(1+t^2) = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \\ L(t+t^2) &= \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Thus, the matrix representation of L with respect to the two given bases is:

$$A = \begin{bmatrix} 4 & 4 & 4 \\ 2 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 4 & 0 \end{bmatrix}.$$

b. Find a basis for the null space of L.

A basis for the null space of A is the single vector $[-1, 0, 1]$. Thus, a basis for the null space of L is the single polynomial $p(t) = -1 + t^2$.

c. Find a basis for the range of L.

A basis for the column space of A is $\{[4, 2, 0, 0], [4, 2, 2, 4]\}$. Thus, a basis for the range of L is

$$\left\{ \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \right\}.$$

(15) 4. Let A be a 3×3 matrix with real entries. Suppose that the numbers 1, 2, -4 are eigenvalues of A, and that $(1,1,1)$, $(1,0,2)$, and $(4,2,3)$ are eigenvectors associated with the respective eigenvalues. Let $\vec{x} = (1,2,3)$.

a. $A\vec{x} = ?$

The easiest way to compute $A\vec{x}$ is to use the result of part c. A second method, is to use the eigenfunction expansion of the vector $\vec{x}$. Thus, from $[1,2,3] = 4[1,1,1] + [1,0,2] - [4,2,3]$, we have $A[1,2,3]^T = 4([1,1,1]) + (2[1,0,2]) - (-4[4,2,3]) = [22,12,20]$.

b. Find a diagonal matrix D, and a nonsingular matrix P, such that $A = PDP^{-1}$.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{bmatrix}. \quad P = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 0 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

c. $A = ?$

$$A = \begin{bmatrix} -12\frac{2}{3} & 6\frac{1}{3} & 7\frac{1}{3} \\ -6\frac{2}{3} & 4\frac{1}{3} & 3\frac{1}{3} \\ -10\frac{2}{3} & 4\frac{1}{3} & 7\frac{1}{3} \end{bmatrix}.$$

(10) 5. Graph the conic section $6x^2 - 8xy + 3y^2 = 4$. Be sure to state whether the conic section is an ellipse, a hyperbola, or a parabola. If an ellipse, find the major axes and the lengths of both axes. If a hyperbola, find the major axes and the asymptotic lines, and if it's a parabola, the major axies and the vertex.

The symmetric matrix which comes from the above quadratic form is $A = \begin{bmatrix} 6 & -4 \\ -4 & 3 \end{bmatrix}$. The eigenvalues of this matrix and an associated set of orthonormal eigenvectors are $\lambda_1 \approx 0.228$, $\lambda_2 \approx 8.772$, $\mu_1 \approx [0.569, 0.822]$, and $\mu_2 \approx [0.822, -0.569]$. Since both eigenvalues are positive, the quadratic form must be an ellipse. Writing a vector in terms of the orthonormal basis we have $\vec{x} = x_1\mu_1 + x_2\mu_2$. Thus, we have the following equation for the scalars x_1 and x_2: $\lambda_1 x_1^2 + \lambda_2 x_2^2 = 4$. Some algebraic manipulation leads to:

$$\frac{x_1^2}{(4.189)^2} + \frac{x_2^2}{(.675)^2} \approx 1\,.$$

From this equation we see that the length of the major axis of the ellipse is $\approx 2(4.189)$, this major axis is parallel to the direction of the eigenvector $[0.569, 0.822]$; the direction of the minor axis is parallel to the direction of the eigenvector $[0.822, -0.569]$, and its length is $\approx 2(0.675)$.

(15) 6. The questions below refer to the following system of equations:

$$\begin{aligned} 2x_1 + x_2 &= 11 \\ x_1 + 2x_2 &= 1 \\ -5x_1 + 7x_2 &= -56 \end{aligned}$$

a. Does the above system have a solution. If yes, find it, if not, how do you know this?

The system has a unique solution: $x_1 = 7$, and $x_2 = -3$.

b. Assume that the system does not have a solution. Find the least squares solution by using a QR factorization of the coefficient matrix of the system.

The matrix Q has as its columns an orthonormal basis of the column space of A. Thus, $Q \approx \begin{bmatrix} 0.365 & 0.654 \\ 0.183 & 0.647 \\ -0.913 & 0.391 \end{bmatrix}$. The matrix R equals $Q^T A \approx \begin{bmatrix} 5.477 & -5.660 \\ 0 & 4.687 \end{bmatrix}$.
Solving the reduced normal equations $R\vec{x} = Q^T\vec{b}$, we get $x_1 = 7$ and $x_2 \approx -2.999$.