1. (15) Define the following:

   (a) basis of a vector space

   A basis of a vector space is a set of vectors this is linearly independent and spanning.

   (b) null space of an $m \times n$ matrix A

   The null space of an $m \times n$ matrix A is

$$\left\{ \vec{x} \in R^n : A\vec{x} = \vec{0} \right\} .$$

   (c) linear transformation

   A linear transformation L is a function from one vector space V into another vector space W such that for all $\vec{x}$, $\vec{y} \in V$ and scalars α and β we have

$$L(\alpha\vec{x} + \beta\vec{y}) = \alpha L(\vec{x}) + \beta L(\vec{y}) .$$

2. (20) Let P_4 be the vector space of polynomials with real coefficients of degree 3 or less. Let B designate the basis $\left\{t - 1,\, t^2,\, t^3 + 2t,\, t^2 + t\right\}$ of this vector space.

   (a) If the coordinates of $\vec{p} \in P^4$ with respect to the basis B are $[2,\ -1,\ 3,\ 1]$, what is $\vec{p}$?

$$\begin{aligned} \vec{p} &= 2(t-1) - t^2 + 3\left(t^3 + 2t\right) + \left(t^2 + t\right) \\ &= 3t^3 + 9t - 2 \end{aligned}$$

   (b) Let $S = \left\{1,\, t,\, t^2,\, t^3\right\}$. Find the change of basis matrix, A, that converts coordinates with respect to S into coordinates with respect to B. That is,find A such that

$$[\vec{p}]_B = A\,[\vec{p}]_S$$

   The easiest way to do this is to find the matrix Q that changes coordinates with respect to B into coordinates with respect to S, and then use the fact that $A = Q^{-1}$. The matrix Q equals

$$Q = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

   and

$$A = Q^{-1} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & -2 \end{bmatrix}$$

3. (30) Let $A = \begin{bmatrix} 4 & 4 & -3 & 2 \\ 2 & 2 & -1 & 2 \\ 3 & 3 & -2 & 2 \end{bmatrix}$.

(a) Find a basis for the null space of A.

The first thing to do is to find the reduced row echelon form of the matrix A. It is

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

The reduced row echelon form of A tells us that if $(x_1, x_2, x_3, x_4) \in NS(A)$ then we must have $x_1 = -x_2 - 2x_4$ and $x_3 = -2x_4$. Thus, a basis for the null space of A is

$$\{(-1, 1, 0, 0), (-2, 0, -2, 1)\} .$$

(b) Find a basis for the row space of A.

A basis for the row space of A is

$$\{(1, 1, 0, 2), (0, 0, 1, 2)\} .$$

(c) Find a basis for the column space of A.

A basis for the column space of A is

$$\{(4, 2, 3), (-3, -1, -2)\} .$$

4. (25) Let $A = \begin{bmatrix} 4 & 4 & -3 & 2 \\ 2 & 2 & -1 & 2 \\ 3 & 3 & -2 & 2 \end{bmatrix}$. This matrix represents a linear transformation $L : M_{2,2} \to P_3$ with respect to the bases

$S = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ and $T_1 = \{1, t, t^2\}$ of $M_{2,2}$ and P_3 respectively.

(a) Compute $L\left(\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\right)$.

The coordinates of $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ with respect to the basis S are $[2, 0, 1, -1]$, and the coordinates of $L\left(\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\right)$ with respect to the basis T_1 are

$$A \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} .$$

Thus, $L\left(\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\right) = 3 + t + 2t^2$.

(b) Find a basis for the kernel of L.

Since we have a basis for the null space of A, we can use it to construct a basis for the kernel of L. Thus, a basis for the kernel of L consists of the following two matrices

$$-\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \text{ and}$$

$$-2\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - 2\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -2 & -2 \end{bmatrix}$$

(c) Find a basis for the image of L.

Since the column space of A consists of the coordinates of vectors in the image of L, we can use a basis of the column space of A to give us a basis for the image of L. So, a basis of the image of L is

$$\left\{4 + 2t + 3t^2, \; -3 - t - t^2\right\} .$$

(d) What is the matrix representation of L with respect to the bases S and $T_2 = \left\{4 + 2t + 3t^2, \; -3 - t - 2t^2, \, t\right\}$.

We know that $L\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) = 4{+}2t{+}3t^2 = L\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$, and that $L\left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\right) = -3 - t - 2t^2$. Note that these values are the first two vectors in the basis T_2. The value of

$$\begin{aligned} L\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) &= 2 + 2t + 2t^2 \\ &= 2\left(4 + 2t + 3t^2\right) + 2\left(-3 - t - 2t^2\right) . \end{aligned}$$

Thus, the matrix representation of L with respect to the bases S and T_2 is

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

5. (10) Let V be an n dimensional vector space with basis $S = \{\vec{u}_1,\ \vec{u}_2,\ \cdots,\ \vec{u}_n\}$. Let $[\vec{x}]$ denote the coordinates of $\vec{x} \in V$ with respect to the basis S. Show that the map

$$L : V \to R^n$$

given by $L(\vec{x}) = [\vec{x}]$ is a linear transformation.

To see that mapping a vector onto its coordinates with respect to any basis is fairly easy. First suppose, we have $\vec{x} = x_1\vec{u}_1 + \cdots + x_n\vec{u}_n$ and $\vec{y} = y_1\vec{u}_1 + \cdots + y_n\vec{u}_n$, then

$$\begin{aligned} \vec{x} + \vec{y} &= (x_1\vec{u}_1 + \cdots + x_n\vec{u}_n) + (y_1\vec{u}_1 + \cdots + y_n\vec{u}_n) \\ &= (x_1 + y_1)\,\vec{u}_1 + \cdots + (x_n + y_n)\,\vec{u}_n \end{aligned}$$

From this expression for $\vec{x} + \vec{y}$ we have

$$\begin{aligned} [\vec{x} + \vec{y}] &= (x_1 + y_1,\ \cdots,\ x_n + y_n) \\ &= (x_1,\ \cdots,\ x_n) + (y_1,\ \cdots,\ y_n) \\ &= [\vec{x}] + [\vec{y}]\ . \end{aligned}$$

This translates into

$$L(\vec{x} + \vec{y}) = L(\vec{x}) + L(\vec{y})\ .$$

To verify the second property of a linear transformation we have

$$\begin{aligned} \alpha\vec{x} &= a\,(x_1\vec{u}_1 + \cdots + x_n\vec{u}_n) \\ &= ax_1\vec{u}_1 + \cdots + ax_n\vec{u}_n\ . \end{aligned}$$

This implies

$$\begin{aligned} [\alpha\vec{x}] &= [ax_1\vec{u}_1 + \cdots + ax_n\vec{u}_n] \\ &= (ax_1, \cdots,\ ax_n) \\ &= a\,(x_1, \cdots,\ x_n) = \alpha\,[\vec{x}]\ , \end{aligned}$$

and in terms of L this says

$$L(\alpha\vec{x}) = \alpha L(\vec{x})\ .$$

Thus, L is a linear transformation.