1. (60) Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

Before answering any of the questions below a row echelon form of A is calcuated. It is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for the null space of A.

From the matrix above we see that the rank of A equals 3, and hence the dimension of its null space is 0. Thus, the null space of A is just the $\vec{0}$ vector, and there is no basis.

(b) Find a basis for the row space of A.

The row space of A is a subspace of R^3, and in fact it is all of R^3. A basis for this subspace is $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$.

(c) Find an orthonormal basis for the column space of A. Be sure to explain exactly what you are doing in constructing the orthonormal basis.

A basis for the column space of A is given by the three colums of A, and they are

$$\vec{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \vec{c}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{c}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Using this basis, the Gram-Schmidt algorithm is used to construct an orthonormal basis of the column space of A.

$$\begin{aligned} \vec{u}_1 &= \frac{\vec{c}_1}{\|\vec{c}_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \\ \vec{u}_2 &= \frac{\vec{c}_2 - [\langle \vec{c}_2, \vec{u}_1 \rangle \vec{u}_1]}{\|\vec{c}_2 - [\langle \vec{c}_2, \vec{u}_1 \rangle \vec{u}_1]\|} = \begin{bmatrix} 1/\sqrt{6} \\ \sqrt{6}/3 \\ 0 \\ -1/\sqrt{6} \end{bmatrix} \\ \vec{u}_3 &= \frac{\vec{c}_3 - [\langle \vec{c}_3, \vec{u}_1 \rangle \vec{u}_1 + \langle \vec{c}_3, \vec{u}_2 \rangle \vec{u}_2]}{\|\vec{c}_3 - [\langle \vec{c}_3, \vec{u}_1 \rangle \vec{u}_1 + \langle \vec{c}_3, \vec{u}_2 \rangle \vec{u}_2]\|} = \begin{bmatrix} -\sqrt{3}/3 \\ \sqrt{3}/3 \\ 0 \\ \sqrt{3}/3 \end{bmatrix} \end{aligned}$$

The set $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthonormal basis of the column space of A.

A simpler method to obtain an orthonormal basis for the column space of A is to observe that

$$\text{CS}(A) = \{\vec{x} = (x_1, x_2, x_3, x_4) : x_3 = 0\}.$$

Thus, another orthonormal basis for the column space of A is the set

$$\{\vec{e}_1, \vec{e}_2, \vec{e}_4\}.$$

(d) Explain under what conditions the equation $A\vec{x} = \vec{b}$ can be solved.

There are several equivalent conditions that ensure the solvability of this equation. One of them is that the vector $\vec{b}$ belongs to the column space of A.

(e) Find the least squares solution of the equation $A\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Be sure to explain each step.

Since every vector in the column space of A must have a zero in its third slot, we see that the vector $\vec{b} = (1,\ 1,\ 1,\ 1)^{\mathrm{T}}$ is not in the column space of A. Hence the equation does not have a solution. The least squares solution of this problem is the solution to the equation

$$A\vec{x} = \mathrm{Proj}_{\mathrm{CS}(A)}\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

The solution to this problem is

$$\vec{x} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}.$$

This solution was found directly without resorting to the normal equations or a Q-R factorization of the matrix A.

2. (10) Suppose V is a vector space with inner product denoted by $\langle \cdot, \cdot \rangle$, and that the set $U = \{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ is an orthonormal basis of V with respect to this inner product.

(a) If $\vec{x} \in V$, show that the coordinates of $\vec{x}$ with respect to the basis U are $[x_1, x_2, x_3, x_4]$, where $x_i = \langle \vec{x}, \vec{u}_i \rangle$ for $i = 1, \cdots, 4$.

Suppose that $\vec{x} = x_1\vec{u}_1 + x_2\vec{u}_2 + x_3\vec{u}_3 + x_4\vec{u}_4$. Then we have

$$\begin{aligned} \langle \vec{x}, \vec{u}_1 \rangle &= \langle x_1\vec{u}_1 + x_2\vec{u}_2 + x_3\vec{u}_3 + x_4\vec{u}_4, \vec{u}_1 \rangle \\ &= x_1 \langle \vec{u}_1, \vec{u}_1 \rangle + x_2 \langle \vec{u}_2, \vec{u}_1 \rangle + x_3 \langle \vec{u}_3, \vec{u}_1 \rangle + x_4 \langle \vec{u}_4, \vec{u}_1 \rangle \\ &= x_1 \cdot 1 + x_2 \cdot 0 + x_3 \cdot 0 + x_4 \cdot 0 \\ &= x_1. \end{aligned}$$

A similar argument readily shows that $\langle \vec{x}, \vec{u}_2 \rangle = x_2$, $\langle \vec{x}, \vec{u}_3 \rangle = x_3$, and $\langle \vec{x}, \vec{u}_4 \rangle = x_4$.

(b) If $\vec{x}$ and $\vec{y}$ are two vectors in V, show that their inner product is equal to

$$\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{4} x_i y_i,$$

where x_i and y_i denote the coordinates of $\vec{x}$ and $\vec{y}$ respectively with respect to the orthonormal basis U.

$$\begin{aligned} \langle \vec{x}, \vec{y} \rangle &= \left\langle \sum_{i=1}^{4} x_i \vec{u}_i, \sum_{j=1}^{4} y_j \vec{u}_j \right\rangle \\ &= \sum_{i=1}^{4} \sum_{j=1}^{4} x_i y_j \langle \vec{u}_i, \vec{u}_j \rangle \\ &= \sum_{i=1}^{4} \sum_{j=1}^{4} x_i y_j \delta_{i,j} \\ &= \sum_{i=1}^{4} x_i y_i. \end{aligned}$$

3. (35) Let V be the vector space of all polynomials of degree 2 or less. That is, $V = P_3$. Let W be R^4. Define the mapping $L{:}V \to W$ by

$$L(\vec{p}) = \left(p(-1),\, p(0),\, p(1),\, \int_{-1}^{1} p(t)\ dt\right).$$

(a) Verify that L is a linear transformation.

Let $\vec{p}$ and $\vec{q}$ be any two polymonials in V, and let α be any scalar. Then we have

$$\begin{aligned} L(\alpha\vec{p}+\vec{q}) &= \left(\alpha p(-1)+q(-1),\, \alpha p(0)+q(0),\, \alpha p(1)+q(1),\, \int_{-1}^{1} (\alpha p(t)+q(t))\ dt\right) \\ &= \alpha\left(p(-1),\, p(0),\, p(1),\, \int_{-1}^{1} p(t)\ dt\right) + \left(q(-1),\, q(0),\, q(1),\, \int_{-1}^{1} q(t)\ dt\right) \\ &= \alpha L(\vec{p}) + L(\vec{q}). \end{aligned}$$

(b) Using the standard bases of V and W ($\{1, t, t^2\}$ and $\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\}$ respectively) find the matrix representation of L with respect to these bases.

$$\begin{aligned} L(1) &= (1,\, 1,\, 1,\, 2) \\ L(t) &= (-1,\, 0,\, 1,\, 0) \\ L(t^2) &= (1,\, 0,\, 1,\, 2/3). \end{aligned}$$

Thus, the matrix representation of L with respect to the given bases is

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 2/3 \end{bmatrix}.$$

(c) What is the rank of the matrix you found in part b, and what information does it tell you about kernel and image of the linear transformation L.

The reduced row echelon form of the matrix from part b is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, the rank of the matrix is 3, and the dimension of the null space is 0. This tells us that the kernel of L is just the zero vector, and we also know that the dimension of the image of L must be 3.

4. (15) Suppose A is a 3×3 matrix with the following eigenvalues and associated eigenvectors

λ	$\vec{x}_\lambda$
$-1/2$	$(1,2,-1)$
$3/4$	$(2,5,1)$
1	$(1,2,1)$

If $\vec{x} = (7,16,3)^{\mathrm{T}}$, compute $A^{999}\vec{x}$.

The eigenvectors of A form a basis of R^3. Writing $\vec{x}$ as a linear combination of these eigenvectors we have

$$(7,16,3) = (1,2,-1) + 2\,(2,5,1) + 2\,(1,2,1)\,.$$

Thus,

$$\begin{aligned} A^{999}\vec{x} &= \left(\frac{-1}{2}\right)^{999}(1,2,-1) + 2\left(\frac{3}{4}\right)^{999}(2,5,1) + 2\,(1)^{999}\,(1,2,1) \\ &\approx 2\,(1,2,1)\,. \end{aligned}$$

5. (10) Let A be an $n \times n$ matrix. Explain what an eigenvalue and eigenvector of A are.

An eigenvalue of a matrix A is a number λ for which there is a non-zero vector, $\vec{x} \in R^n$, called an eigenvector such that

$$A\vec{x} = \lambda\vec{x}\ .$$

This *non-zero* vector $\vec{x}$ is said to be an eigenvector of A associated with the eigenvalue λ.

6. (20) Let $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$.

(a) Find a diagonal matrix D and a nonsingular matrix P such that

$$A = PDP^{-1}.$$

The eigenvalues of A with associated eigenvectors are:

$$\begin{aligned} \lambda_1 &= 1, \vec{x}_1 = [1, \ -2] \\ \lambda_2 &= 4, \vec{x}_2 = [1, 1]. \end{aligned}$$

Set $D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$. Then we have

$$A = PDP^{-1}.$$

(b) Find a matrix B such that $B^2 = A$.

It is easy to find a matrix d such that $d^2 = D$. One such matrix is

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Set $B = PdP^{-1}$. Then we have

$$\begin{aligned} B^2 &= \left(PdP^{-1}\right)^2 \\ &= \left(PddP^{-1}\right) \\ &= PDP^{-1} \\ &= A. \end{aligned}$$

Computing PdP^{-1} we have

$$\begin{aligned} B &= PdP^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \\ &= \begin{bmatrix} \frac{5}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{4}{3} \end{bmatrix}. \end{aligned}$$

A quick check verifys that B^2 does indeed equal A.

$$\begin{aligned} B^2 &= \begin{bmatrix} \frac{5}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{4}{3} \end{bmatrix} \begin{bmatrix} \frac{5}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{4}{3} \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \end{aligned}$$