1. (35) Let $V = \mathrm{sp}\{1 + t^2, t + t^3\}$, and consider V as a subspace of P_4.

a. What is the dimension of V?

Since the vectors $1 + t^2$ and $t + t^3$ are linearly independent and span V, they are a basis of V. Hence V has dimension 2.

b. Show that $B_2 = \{2 + t + 2t^2 + t^3, -1 + 3t - t^2 + 3t^3\}$ is a basis of V.

The first thing to observe is that the two vectors in B_2 are linear combinations of the vectors used to define V.

$$2 + t + 2t^2 + t^3 = 2(1 + t^2) + (t + t^3)$$

$$-1 + 3t - t^2 + 3t^3 = -1(1 + t^2) + 3(t + t^3) .$$

Thus, B_2 is a subset of V. Moreover, they are linearly independent. Since there are two linearly independent vectors in B_2 and $\dim(V) = 2$, the set B_2 must be a basis of V.

c. Find the coordinates of $2 + t + 2t^2 + t^3$ with respect to the basis $B_1 = \{1 + t^2, t + t^3\}$.

This was acutally done in part a. From $2 + t + 2t^2 + t^3 = 2(1 + t^2) + (t + t^3)$ we see that the coordinates of $2 + t + 2t^2 + t^3$ with respect to the basis B_1 are $[2, 1]$.

d. Find the change of basis matrix P such that $[\vec{x}]_{B_1} = P[\vec{x}]_{B_2}$.

The columns of P are the coordinates of the vectors in B_2 with respect to the basis B_1. From part a. we have

$$P = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}.$$

2. (25) Let $A = \begin{bmatrix} 1 & 1 & 3 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 4 & 1 \end{bmatrix}$.

a. What is the reduced row echelon form of A?

The reduced row echelon form of A equals

$$rref(A) = \begin{bmatrix} 1 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

b. Find a basis for the column space of A.

Since the first and third columns of the reduced row echelon form of A are linearly independent and the matrix has rank 2. These two columns form a basis for the column space of $rref(A)$. Thus, the first and third columns of A are a basis for the column space of A. That is,

$$\text{a basis for C}(A) \text{ is } \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \right\}.$$

c. Find a basis for the null space of A.

The reduced row echelon form of A tells us that $x_1 = -x_2 - \frac{3}{2}x_4$, and $x_3 = \frac{1}{2}x_4$. Thus, x_2 and x_4 are free variables and x_1 and x_3 are bound variables. Thus, a basis for the null space of A is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3/2 \\ 0 \\ 1/2 \\ 1 \end{bmatrix} \right\}$$

3. (20) Suppose A of problem 2 is the matrix representation of a linear transformation $L : P_4 \to R^3$ with respect to the bases

$$B_1 = \{1 - t, t + t^2, 3t, t^3 - t^2\} \text{ and } B_2 = \{(1, 0, 1), (-1, 1, 1), (2, 1, 1)\}$$

of P_4 and R^3 respectively.

a. What is $L(1 + 3t + t^3)$?

The first step is to find the coordinates of $1 + 3t + t^3$ with respect to B_1. A quick check shows that

$$1 + 3t + t^3 = (1 - t) + (t + t^2) + (3t) + (t^3 - t^2) .$$

Thus, the coordinates are $\left[1, 1, 1, 1\right]$. Using A we see that the coordinates of $L(1 + 3t + t^3)$ with respect to the basis B_2 are

$$\begin{bmatrix} 1 & 1 & 3 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 9 \end{bmatrix} .$$

Thus,

$$L(1 + 3t + t^3) = 5\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 4\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + 9\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 19 \\ 13 \\ 18 \end{bmatrix} .$$

b. Find a basis for the kernel of L.

Since a basis for the null space of A is

$$\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3/2 \\ 0 \\ 1/2 \\ 1 \end{bmatrix}\right\},$$

a basis for the kernel of L is

$$\left\{-(1 - t) + (t + t^2),\ -3/2(1 - t) + 1/2(3t) + (t^3 - t^2)\right\} \text{ or}$$

$$\left\{t^2 + 2t - 1, t^3 - t^2 + 3t - \frac{3}{2}\right\} .$$

c. Find a basis for the range of L.

Since a basis for the column space of A is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \right\},$$

a basis for the image (range) of L is

$\{(1, 0, 1) + (-1, 1, 1) + 2(2, 1, 1), 3(1, 0, 1) + (-1, 1, 1) + 4(2, 1, 1)\}$ or

$\{(4, 3, 4), (10, 5, 8)\}$.

4. (20) Let A be an $m \times n$ matrix, with $n \geq 3$. Let $\vec{a}_i$ denote the i^{th} column of A. Suppose that $2\vec{a}_1 + \vec{a}_3 = \vec{0}$.

a. Find at least one non-zero vector in the null space of A.

If $\vec{x} \in R^n$, then $A\vec{x} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$, where x_i is the i^{th} component of $\vec{x}$, and $\vec{a}_i$ is the i^{th} column of A. So if we set $\vec{x} = (2, 0, 1, \cdots)$, then we have

$$A\vec{x} = 2\vec{a}_1 + \vec{a}_3 = \vec{0},$$

and $\vec{x} \in N(A)$

b. Let $\vec{b}_i$ denote the i^{th} column of a matrix B, which is row equivalent to A.Show that $2\vec{b}_1 + \vec{b}_3 = \vec{0}$.

If B is row equivalent to A, then there is a sequence of elementary row matrices E_i, $1 \leq i \leq k$ such that

$$E_1 \cdots E_k A = B.$$

Let $\vec{x} = (2, 0, 1, \cdots)$. Then we have

$$\begin{aligned} B\vec{x} &= (E_1 \cdots E_k A)\vec{x} \\ &= E_1 \cdots E_k (A\vec{x}) \\ &= E_1 \cdots E_k (\vec{0}) = \vec{0}. \end{aligned}$$

Thus, 2 times the first row of B plus the second row equals $\vec{0}$.