1. (60) Let $A = \begin{bmatrix} 1 & -1 & 3 & -2 \\ 1 & 0 & 4 & -2 \\ 2 & -2 & 6 & -4 \\ 1 & 0 & 4 & -2 \end{bmatrix}$.

a. Find a basis for the null space of A.

A is row equivalent to the matrix

$$B = \begin{bmatrix} 1 & 0 & 4 & -2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

From B we see that the rank of A is 2, and hence the dimension of A's null space is also 2. A basis for the null space is

$$\left\{ \begin{bmatrix} -4 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

b. Find a basis for the row space of A.

A basis for the row space of A is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

c. Find an orthonormal basis for the column space of A.

A basis for the column space of A is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 0 \end{bmatrix} \right\}.$$

Use Gram-Schmidt to construct an ortho-normal basis.

$$\vec{q}_1 = \frac{(1, 1, 2, 1)}{\sqrt{7}}$$

$$\vec{q}_2 = \frac{(-1, 0, -2, 0) - (-5/\sqrt{7})\dfrac{(1, 1, 2, 1)}{\sqrt{7}}}{\text{length of numerator}}$$

$$= \frac{(-2/7, 5/7, -4/7, 5/7)}{\text{length of numerator}}$$

$$= \frac{(-2, 5, -4, 5)}{\sqrt{70}}.$$

An orthonormal basis of the column space of A is

$$\left\{ \frac{(1, 1, 2, 1)}{\sqrt{7}}, \frac{(-2, 5, -4, 5)}{\sqrt{70}} \right\}$$

d. Explain how to find the least squares solution of the equation $A\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

One way to find the least squares solution is to solve the normal equations

$$A^T A\vec{x} = A^T\vec{b}.$$

A second way is to actually solve the equation $A\vec{x}$ =$\text{Proj}_{C(A)}\vec{b} = \langle \vec{q}_1, \vec{b} \rangle \vec{q}_1 + \langle \vec{q}_2, \vec{b} \rangle \vec{q}_2$. That is,

$$A\vec{x} = \frac{5}{7}(1, 1, 2, 1) + \frac{12}{70}(-2, 5, 4, 5)$$

2. (40) Let $L : R^3 \to R^3$ be a linear transformation. Suppose that

$$L\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, L\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, L\left(\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

a. Find the matrix representation of L with respect to the standard basis of R^3.

Note; the three vectors, $\vec{x}$, for which we know $L(\vec{x})$ form a basis for R^3. Denote this basis by B. Then let $\hat{A}$ denote the matrix representation of L with respect to B and the standard basis, S.

That is,

$$[L(x)]_S = \hat{A}[\vec{x}]_B \text{ with } \hat{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

Let P denote the change of basis matrix between B and S. That is, $[\vec{x}]_S = P[\vec{x}]_B$, and $P = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$. Thus,

$$[L(x)]_S = \hat{A}[\vec{x}]_B = [L(x)]_S = \hat{A}P^{-1}[\vec{x}]_S .$$

Thus, the matrix representation of L with respect to the standard basis is

$$A = \hat{A}P^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

b. Find a basis for the kernel of L.

It's clear that the dimension of the range (image) of L is two, which implies that the dimension of the kernel of L is 1. Since $(2, 1, 3)$ is in the kernel, its span must be the entire kernel. Thus,

$$\{(2, 1, 3)\}$$

is a basis for the kernel of L.

3. (10) Let A be an $n \times n$ matrix. Explain what an eigenvalue and eigenvector of A are.

An eigenvalue of the matrix A is a number λ for which there is a non-zero vector $\vec{x}$ such that

$$A\vec{x} = \lambda\vec{x}\,.$$

4. (30) Let $A = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

a. Find a diagonal matrix D and a nonsingular matrix P such that

$$A = PDP^{-1}.$$

The eigenvalues and eigenvectors of A are:

λ	$\vec{x}_\lambda$
1	$(1, 1)$
1/4	$(-1/2, 1)$

.

Thus, $P = \begin{bmatrix} -1/2 & 1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1/4 & 0 \\ 0 & 1 \end{bmatrix}$ is one possible choice for P and D.

b. Find a matrix B such that $B^2 = A$.

Since the matrix $C = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$ satisfies $C^2 = D$, the matrix $B = PCP^{-1}$ is a square root of A.

$$\begin{aligned} B &= \begin{bmatrix} -1/2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1/2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \end{aligned}$$

c. Let $\vec{x} = \begin{bmatrix} 1/2 \\ 3 \end{bmatrix}$. Determine $\lim\limits_{n\to\infty} A^n\vec{x}$.

The vector $\vec{x} = (1/2, 3) = \frac{5}{3}(-1/2, 1) + \frac{4}{3}(1, 1)$. Thus,

$$\begin{aligned} \lim_{n\to\infty} A^n\vec{x} &= \lim_{n\to\infty} A^n\left(\tfrac{5}{3}(-1/2, 1) + \tfrac{4}{3}(1, 1)\right) \\ &= \lim_{n\to\infty} A^n\left[\tfrac{5}{3}(-1/2, 1)\right] + \lim_{n\to\infty} A^n\left[\tfrac{4}{3}(1, 1)\right] \\ &= \tfrac{5}{3}\lim_{n\to\infty} A^n\left[(-1/2, 1)\right] + \tfrac{4}{3}\lim_{n\to\infty} A^n\left[(1, 1)\right] \\ &= \tfrac{5}{3}\left(\tfrac{1}{4}\right)^n(-1/2, 1)\tfrac{4}{3}\lim_{n\to\infty}(1)^n\left[(1, 1)\right] \\ &= \tfrac{4}{3}(1, 1)\,. \end{aligned}$$

5. (10) The matrix *A* below has rank 3 and is in reduced row echelon form. Fill in the missing entries.

$$\begin{bmatrix} 1 & 2 & \mathbf{0} & 6 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & 3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$