1. (20) Suppose the function $y(x)$ satisfies the differential equation:

$$\frac{dy}{dx} = 2y(15 - y)\ .$$

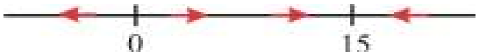
a. State the existence/uniqueness theorem for first order differential equations. For which initial conditions does this theorem apply to the above differential equation.

For an equation of the form $\frac{dy}{dx} = f(x,y)$ with initial condition $y(x_0) = y_0$, if there is an open rectangle R, containing the point (x_0, y_0), on which the function f and its partial derivative with respect to y are both continuous, then there is $\delta > 0$ such that on the interval $(x_0 - \delta, x_0 + \delta)$ there is exactly one solution to the initial value problem.

For the differential equation above $f(x, y) = 2y(15 - y)$ and $\frac{\partial f}{\partial y} = 30 - 4y$. Since both of these functions are continuous for all values of x and y, the existence/uniqueness theorem applies to all initial conditions.

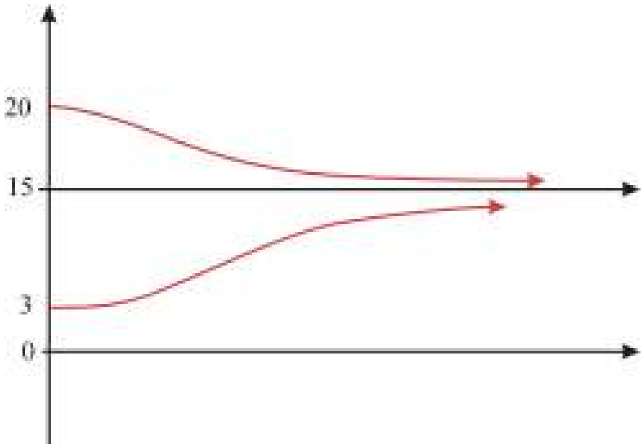
b. Draw the phase line for this equation. Be sure to locate all equilibrium points.

There are only two equilibrium points. $y = 0$ and $y = 15$. The phase line is shown below.



c. Sketch solution curves in the x-y plane for the following initial conditions:

$$y(0) = 0, y(0) = 3, y(0) = 15, y(0) = 20\ .$$



d. Explain why none of the curves you drew in part c. can cross each other.

If two of these curves crossed at the point (x_0, y_0) then since they are curves of solutions to the differential equation and both of them satisfy the initial condition $y(x_0) = y_0$ the existence and uniqueness theorem says these two solutions have to be the same, which means there curves have to be the same, but the curves are not the same. so they can't cross.

e. Give an argument to justify the statement: if $y(0)$ is positive and $y(x)$ satisfies the differential equation above, then $\lim_{x\to\infty} y(x) = 15$.

I argue for the case $y(0)$ lies strictly between 0 and 15. The argument is similar for $y(0) > 15$, and if $y(0) = 15$, then $y(x) \equiv 15$. So if $y(0)$ lies between 0 and 15, then $y(x)$ must always lie between 0 and 15, from part c. Thus, it's derivative is always positive and y must be an increasing function. Since $y(x)$ is bounded above by 15, we must have the limit of $y(x)$ existing as x goes to infinity. Suppose this limiting value is less than 15; call it L. Then the quantity $2y(15-y) > 2y(0)(15-L) > 0$. That is, dy/dx is bounded away from 0, and this means that after a finite amount of time the value of $y(x)$ must exceed 15, a contradiction. So L must equal 15.

2. (30) Each of the questions below refer to the differential equation

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 13y = \sin(t)\ .$$

a. Is this a linear equation?

The differential operator associated with this equation is $L[f] = \dfrac{d^2f}{dt^2} - 4\dfrac{df}{dt} + 13f$. One quickly checks that this operator satisfies the two equations:

$$L(f_1 + f_2) = L(f_1) + L(f_2)$$
$$L(\alpha f) = \alpha L(f)\ ,$$

where f_1, f_2, and f are arbitrary functions with two derivatives, and α is any constant. Thus, L is a linear operator, which means the differential equation is linear.

b. Rewrite this equation as a system of first order equations.

Setting $x_1 = y$ and $x_2 = dy/dt$ we have

$$\begin{aligned}\frac{dx_1}{dt} &= x_2\\ \frac{dx_2}{dt} &= \frac{d^2x_1}{dt^2} = \frac{d^2y}{dt^2} = 4\frac{dy}{dt} - 13y + \sin(t)\\ &= -13x_1 + 4x_2 + \sin(t)\ .\end{aligned}$$

c. Find a fundamental solution set for this equation.

The auxiliary equation for this second order differential equation is:

$$0 = r^2 - 4r + 13\ .$$

Thus,

$$r = \frac{4 \pm \sqrt{16-52}}{2} = 2 \pm 3i$$

Thus, taking the real and imaginary part of the solution $e^{(2+3i)t}$, we have the following two solutions

$$e^{2t}\cos 3t \text{ and } e^{2t}\sin 3t\ .$$

Computing their Wronskian at $t = 0$ we have

$$\det\left[\begin{array}{cc} e^{2t}\cos 3t & e^{2t}\sin 3t \\ 2(\cos 3t)e^{2t} - 3e^{2t}\sin 3t & 3(\cos 3t)e^{2t} + 2e^{2t}\sin 3t \end{array}\right]\Bigg|_{t=0} = \det\left[\begin{array}{cc} 1 & 0 \\ 2 & 3 \end{array}\right]$$
$$= 3\ .$$

Since the Wronskian is non-zero, these two functions form a fundamental solution set.

d. Find the general solution to this equation.
The general solution to this equation has the form

$$y = c_1e^{2t}\cos 3t + c_2e^{2t}\sin 3t + y_p\ ,$$

where y_p is a particular solution. Since the forcing term is $\sin t$, we look for a y_p of the form

$$y_p = a\sin t + b\cos t\ .$$

Calculating $L[y_p]$ we have

$$\begin{aligned} 13y_p &= 13(a\sin t + b\cos t) \\ (-4)y_p' &= (-4)(a\cos t - b\sin t) \\ y_p'' &= -a\sin t - b\cos t\ . \end{aligned}$$

Thus,

$$\begin{aligned} L(y) &= (-a\sin t - b\cos t) - 4(a\cos t - b\sin t) + 13(a\sin t + b\cos t) \\ &= (-4a + 12b)\cos t + (12a + 4b)\sin t\ . \end{aligned}$$

Since $L(y_p)$ must equal $\sin t$, we must have $a = 3/40$ and $b = 1/40$. Thus, the general solution is

$$y = c_1e^{2t}\cos 3t + c_2e^{2t}\sin 3t + \frac{3}{40}\sin t + \frac{1}{40}\cos t\ .$$

e. Find a solution to this equation which also satisfies the initial conditions:

$$y(0) = 1,\ y'(0) = 1\ .$$

We need to determine values of c_1 and c_2 in the general solution that will force y to satisfy the initial conditions. This leads to the equations

$$\begin{aligned} 1 = y(0) &= c_1 + \frac{1}{40} \\ 1 = y'(0) &= \frac{3}{40} + 2c_1 + 3c_2\ . \end{aligned}$$

Thus, $c_1 = 39/40$ and $c_2 = -41/120$, and the solution to the initial value problem is

$$y = \frac{39}{40}e^{2t}\cos 3t - \frac{41}{120}e^{2t}\sin 3t + \frac{3}{40}\sin t + \frac{1}{40}\cos t$$

3. (30) Let $A = \begin{bmatrix} 3 & -5 \\ -1 & -1 \end{bmatrix}$.

a. Find the eigenvalues and eigenvectors of this matrix.

To find the eigenvalues of A, solve the equation

$$\det(A - \lambda I) = \det\begin{bmatrix} 3-\lambda & -5 \\ -1 & -1-\lambda \end{bmatrix} = (\lambda+1)(\lambda-3) - 5$$

$$= \lambda^2 - 2\lambda - 8 = (\lambda - 4)(\lambda + 2) = 0\,.$$

The eigenvalues are –2 and 4. The eigenvector for -2, $\vec{x}_{(-2)}$, satisfies the equation

$$\begin{bmatrix} 5 & -5 \\ -1 & 1 \end{bmatrix}\vec{x}_{(-2)} = \vec{0}\,.$$

So $\vec{x}_{(-2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The eigenvalue for 4, $\vec{x}_4$, satisfies the equation

$$\begin{bmatrix} -1 & -5 \\ -1 & -5 \end{bmatrix}\vec{x}_4 = \vec{0}\,.$$

So $\vec{x}_4 = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$

b. Sketch the phase plane for the following system of differential equations:

$$\frac{dx_1}{dt} = 3x_1 - 5x_2$$

$$\frac{dx_2}{dt} = -x_1 - x_2\,.$$

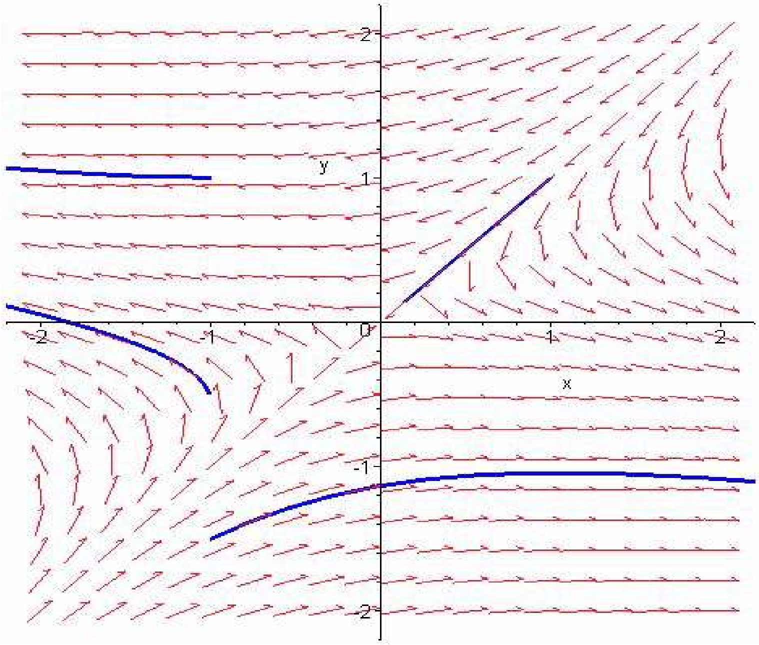
Be sure to indicate on your sketch the asymptotic behavior of solutions as $t \to \infty$.

The general solution of this system is

$$y = c_1 e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} -5 \\ 1 \end{bmatrix}$$

$$= e^{4t}\left(c_1 e^{-6t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -5 \\ 1 \end{bmatrix}\right)$$

From the two eigenvectors we know that if a solution starts on either of the lines $y = x$ or $y = -x/5$, they will stay on that respective line. We also see from the last displayed line above that all solutions are asymptotic to the line $y = -x/5$. Moreover, if a solution does not start on the line $y = x$, it blows up, while if it starts on this line, it converges to $(0, 0)$ as $t \to \infty$. A phase plane plot is shown below



The blue lines are plots of solutions with the following initial conditions: $\left[1,\ 1\right]$, $\left[-1,\ 1\right]$, $\left[-1,\ -1/2\right]$, $\left[-1,\ -3/2\right]$.

c. Find the solution to the initial value problem:

$$\frac{dx_1}{dt} = 3x_1 - 5x_2\ ,\ x_1(0) = 1$$
$$\frac{dx_2}{dt} = -x_1 - x_2\ ,\ x_2(0) = 3\ .$$

The general solution to this system is

$$\vec{x}(t) = c_1 e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} -5 \\ 1 \end{bmatrix}.$$

Setting $t = 0$, we have

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -5 \\ 1 \end{bmatrix}.$$

The solution to this is: $c_1 = 8/3$ and $c_2 = 1/3$. Thus, the solution to the initial value problem is

$$\vec{x}(t) = \frac{8}{3}e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{3}e^{4t}\begin{bmatrix} -5 \\ 1 \end{bmatrix}.$$

4. (10) Suppose that L is a linear operator. Explain why every solution to the equation

$$L[X] = f,$$

has the form $X = X_g + X_p$, where X_g is any solution to the homogeneous equation $L(X) = 0$, and X_p is a particular solution to the given non-homogeneous equation.

First we show that if $X = X_g + X_p$, then X solves the equation.

$$\begin{aligned} L[X_g + X_p] &= L[X_g] + L[X_p] \\ &= 0 + f = f. \end{aligned}$$

Conversely suppose X_p is a particular solution and X is any other solution. Then we have

$$\begin{aligned} L(X - X_p) &= L[X] - L[X_p] \\ &= f - f = 0\,. \end{aligned}$$

Thus, $X - X_p$ is a solution to the homogeneous equation, call it X_g, and we have

$$X = X_p + X_g\,.$$

5. (10) Suppose the coefficient matrix A of the system $\vec{x}\,' = A\vec{x}$ is a 2×2 matrix with the following eigenvalues and eigenvectors:

λ	$-2 + \sqrt{3}\,i$	$-2 - \sqrt{3}\,i$
$\vec{x}_\lambda$	$\begin{bmatrix} 1 \\ 1 + i \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 - i \end{bmatrix}$

Find two linearly independent real valued solutions of the system of $\vec{x}\,' = A\vec{x}$.

A complex valued solution is

$$\begin{aligned} e^{\left(-2+\sqrt{3}\,i\right)t}\begin{bmatrix} 1 \\ 1 + i \end{bmatrix} &= e^{-2t}\left(\cos\left(\sqrt{3}\,t\right) + i\sin\left(\sqrt{3}\,t\right)\right)\begin{bmatrix} 1 \\ 1 + i \end{bmatrix} \\ &= e^{-2t}\left(\begin{bmatrix} \cos\sqrt{3}\,t \\ \cos\sqrt{3}\,t - \sin\sqrt{3}\,t \end{bmatrix} + i\begin{bmatrix} \sin\sqrt{3}\,t \\ \cos\sqrt{3}\,t + \sin\sqrt{3}\,t \end{bmatrix}\right). \end{aligned}$$

Taking the real and imaginary parts of this complex vector valued function we get two real solutions:

$$\begin{aligned} \vec{x}_1 &= e^{-2t}\begin{bmatrix} \cos\sqrt{3}\,t \\ \cos\sqrt{3}\,t - \sin\sqrt{3}\,t \end{bmatrix} \\ \vec{x}_2 &= e^{-2t}\begin{bmatrix} \sin\sqrt{3}\,t \\ \cos\sqrt{3}\,t + \sin\sqrt{3}\,t \end{bmatrix}. \end{aligned}$$

To verify that they are linearly independent we compute the Wronskian.

$$\begin{aligned} W\left(\vec{x}_1, \vec{x}_2\right) &= \det\begin{bmatrix} e^{-2t}\cos\sqrt{3}\,t & e^{-2t}\sin\sqrt{3}\,t \\ e^{-2t}\left(\cos\sqrt{3}\,t - \sin\sqrt{3}\,t\right) & e^{-2t}\left(\cos\sqrt{3}\,t + \sin\sqrt{3}\,t\right) \end{bmatrix} \\ &= \left(\cos^2\sqrt{3}\,t\right)e^{2(-2t)} + \left(\sin^2\sqrt{3}\,t\right)e^{2(-2t)} \\ &= e^{-4t}\,. \end{aligned}$$

Since the Wronskian is never zero, these solutions are linearly independent.