1. (50) Consider the following initial value problem:

$$\frac{d^2y}{dt^2} + 4y = \sin t$$

$$y(0) = 1, y'(0) = 2 .$$

a. What are some adjectives that are applicable to this problem?

Second order, linear, nonhomogeneous.

b. What differential operator is associated with this problem?

$$L[f] = \frac{d^2f}{dt^2} + 4f .$$

c. Is this operator linear?

The operator is linear as the following verifies:

$$L[f+g] = \frac{d^2(f+g)}{dt^2} + 4(f+g) = \frac{d^2f}{dt^2} + \frac{d^2g}{dt^2} + 4f + 4g = L[f] + L[g]$$

$$L[\alpha f] = \frac{d^2(\alpha f)}{dt^2} + 4(\alpha f) = \alpha\left(\frac{d^2f}{dt^2} + 4f\right) = \alpha L[f] .$$

d. Find the general solution to this differential equation.

The general solution to the homogeneous equation is $c_1 \sin 2t + c_2 \cos 2t$. Seeking a particular solution to the full equation of the form $a \sin t + b \cos t$, we have

$$L(a \sin t + b \cos t) = 3a \sin t + 3b \cos t = \sin t .$$

From which we have $a = 1/3$ and $b = 0$. Thus, the general solution to the non-homogeneous equation is

$$y = c_1 \sin 2t + c_2 \cos 2t + \frac{\sin t}{3} .$$

e. Find the solution to this initial value problem.

Using the general solution to the non-homogeneous equation we pick the constants c_1 and c_2 to satisfy the initial conditions.

$$1 = y(0) = c_2$$

$$2 = y'(0) = 2c_1 + \frac{1}{3} .$$

Thus, we have $c_1 = 5/6$, $c_2 = 1$, and

$$y = \frac{5 \sin 2t}{6} + \cos 2t + \frac{\sin t}{3}$$

f. I used the article "the" in the previous question. Is it possible that there is another solution to this initial value problem? Put in lots of detail when you answer this.

To show the uniqueness of this solution write the initial value problem as a system of first order equations, and apply the existence/uniqueness theorem for first order systems to this equivalent problem. To this end, set $x_1 = y$ and $x_2 = dy/dt$. Then

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = -4x_1 + \sin t\,.$$

The right hand side of this system has the form $\vec{F}(x_1, x_2, t) = \begin{bmatrix} x_2 \\ -4x_1 + \sin t \end{bmatrix}$. $\vec{F}$ is continuous as a function of x_1, x_2, and t everywhere. The two partial derivatives $\frac{\partial \vec{F}}{\partial x_1} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$ and $\frac{\partial \vec{F}}{\partial x_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are also continuous everywhere. Thus, we know there is a unigue solution to the problem $\frac{d\vec{x}}{dt} = \vec{F}(\vec{x}, t), \vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. And this means that the original problem has a unique solution for if it had a second solution, then the corresponding initial value problem for this first order system would have two solutions.

2. (30) **a.** Use the definition of the Laplace transform to verify the formula

$$L[t](s) = \frac{1}{s^2}\,.$$

$$L[t](s) = \int_0^\infty e^{-st} t \, dt = \frac{e^{-st}}{-s} t \Big|_{t=0}^{t=\infty} + \int_0^\infty \frac{e^{-st}}{s} dt$$

$$= 0 + \frac{-1}{s^2} e^{-st} \Big|_{t=0}^{t=\infty} = \frac{1}{s^2}\,.$$

This assumes that $\lim_{t\to\infty} e^{-st} = 0$, which is the case if the real part of $s > 0$.

b. Let $f(t) = t$ and $g(t) = u(t) + \delta(t-3)$, where u is the Heaviside step function and δ is Dirac's delta function. Compute the following two values:

$$(f * g)(2) \text{ and } (f * g)(5)\,,$$

where $f * g$ denotes the convolution of f and g.

$$\begin{aligned}
(f * g)(2) &= \int_0^2 f(2-\tau) g(\tau) \, d\tau \\
&= \int_0^2 (2-\tau)(1 + \delta(\tau - 3)) \, d\tau \\
&= \int_0^2 (2-\tau) \, d\tau + \int_0^2 (2-\tau)\delta(\tau-3) \, d\tau \\
&= 2 + 0 = 2\,.
\end{aligned}$$

$$\begin{aligned}
(f * g)(5) &= \int_0^5 f(5-\tau) g(\tau) \, d\tau \\
&= \int_0^5 (5-\tau)(1 + \delta(\tau - 3)) \, d\tau \\
&= \int_0^5 (5-\tau) \, d\tau + \int_0^5 (5-\tau)\delta(\tau-3) \, d\tau \\
&= \frac{25}{2} + (5-3) = \frac{29}{2}
\end{aligned}$$

3. (30) A mass-spring system is governed by the equation

$$y'' + 2y' + 3y = f(t) .$$

Suppose that $y(0) = 0 = y'(0)$, and the forcing function $f(t)$ is zero except for an impulse force of magnitude 10, which occurs at $t = 2$. Using Laplace transforms, find the solution to this problem.

The function $f(t)$ can be written as $10\delta(t-2)$. Taking the Laplace transform of this equation, we get

$$(s^2 + 2s + 3)Y(s) - sy(0) - y'(0) = 10e^{-2s} .$$

Thus,

$$\begin{aligned} Y(s) &= e^{-2s}\frac{10}{(s^2 + 2s + 3)} = e^{-2s}\frac{10}{(s+1)^2 + 2} \\ &= \frac{10}{\sqrt{2}}e^{-2s}\frac{\sqrt{2}}{(s+1)^2 + 2} = \frac{10}{\sqrt{2}}e^{-2s}L\left[\sin\sqrt{2}\,t\right](s+1) \\ &= \frac{10}{\sqrt{2}}e^{-2s}L\left[e^{-t}\sin\sqrt{2}\,t\right](s) \\ &= \frac{10}{\sqrt{2}}L\left[u(t-2)e^{-(t-2)}\sin\left(\sqrt{2}\,(t-2)\right)\right](s) . \end{aligned}$$

Thus, the solution to this initial value problem is

$$\begin{aligned} y(t) &= \frac{10}{\sqrt{2}}u(t-2)e^{-(t-2)}\sin\left(\sqrt{2}\,(t-2)\right) \\ &= \begin{cases} 0, & 0 \le t \le 2 \\ \dfrac{10}{\sqrt{2}}e^{-(t-2)}\sin\left(\sqrt{2}\,(t-2)\right), & t > 2 \end{cases} \end{aligned}$$

4. (20) For the differential equation

$$x^3(x-1)^2(x-2)y'' + x^2(x-1)(x-2)^2y' + (x-2)y = 0\,.$$

a. which points x_0 are singular points of this differential equation?

Since the coeffients of y'', y', and y are all analytic functions, the only possible singularities can occur at values of x for which the coefficient of y'' is zero. So let's look at the terms that arise when the equation is divided by the coefficient of y''.

$$p(x) = \frac{x^2(x-1)(x-2)^2}{x^3(x-1)^2(x-2)} = \frac{x-2}{x(x-1)}$$

$$q(x) = \frac{x-2}{x^3(x-1)^2(x-2)} = \frac{1}{x^3(x-1)^2}\,.$$

Since both p and q have singularities only at $x = 0$ and $x = 1$, these are only two singular points.

b. which of the singular points are regular singular points?

To see if $x = 0$ is a regular singular point we look at

$$\lim_{x\to 0} xp(x) = \lim_{x\to 0}\frac{x-2}{(x-1)} = 2 \text{ and}$$

$$\lim_{x\to 0} x^2q(x) = \lim_{x\to 0}\frac{1}{x(x-1)^2} = \text{does not exist,}$$

hence 0 is not a regular singular point. Taking similar limits at the singular point 1, we have

$$\lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1}\frac{(x-2)}{x} = -1 \text{ and}$$

$$\lim_{x\to 1}(x-1)^2q(x) = \lim_{x\to 1}\frac{1}{x^3} = 1\,.$$

Thus, 1 is a regular singular point.

5. (20) The equation

$$x^2 y'' + xy' + (x^2 - 9)y = 0$$

is Bessel's equation of order 3.

Before answering the questions below, we compute $L[y]$, where L is the differential operator for Bessel's equation of order 3, and $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$.

$$L[y] = x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} + x \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$+ (x^2 - 9) \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$= (r^2 - 9)a_0 x^r + \left((r+1)^2 - 9\right)a_1 x^{r+1} + \sum_{n=2}^{\infty} \left[\left((r+n)^2 - 9\right)a_n + a_{n-2} \right] x^{n+r} .$$

a. What is the indicial equation for this equation about the regular singular point $x_0 = 0$? Be sure to explain why this equation arises.

The indicial equation is $f(r) = r^2 - 9 = 0$, and arises to force the coefficient of x^r to be zero. For Bessel's equation of order 3 the means $r^2 = 9$, or $r = \pm 3$.

b. The method of Frobenius suggests looking for a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r} .$$

Which specific values of r work for this equation?

As mentioned above $r = \pm 3$.

c. Using the largest value of r from part b, what must a_1 equal?

If r is set equal to 3 then the coefficent of x^{1+r} is

$$\left((3+1)^2 - 9\right)a_1 = 7a_1 .$$

The only way this term will be zero is to set $a_1 = 0$.