Be sure to include all work you wish to have graded in your blue book, and to have your answer to question 1 first, followed by the answer to question 2, etc.

1. (10) Define the Laplace transform of a function $u(t)$.

Ans: $\mathcal{L}[u(t)](s) = \int_0^\infty u(t)e^{-st}\,dt.$

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2. (10) Use the definition of the Laplace transform to verify that $\mathcal{L}[e^{at}f(t)](s) = \mathcal{L}[f](s-a)$, where $\mathcal{L}$ denotes the Laplace transform.

Ans:

$$\begin{aligned}\mathcal{L}[e^{at}f(t)](s) &= \int_0^\infty e^{at}f(t)e^{-st}\,dt \\ &= \int_0^\infty f(t)e^{-(s-a)t}\,dt \\ &= \mathcal{L}[f](s-a)\,.\end{aligned}$$

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3. (15) Suppose that the function $x(t)$ satisfies the initial value problem

$$2x'' - x' + 3x = \sin(5t),\ x(0) = 1,\ x'(0) = -2\,.$$

Let $X(s)$ denote the Laplace transform of $x(t)$. Calculate $X(s)$.

Ans: Taking the Laplace transform of the equation we have:

$$2\left(s^2X(s) - sx(0) - x'(0)\right) - (sX(s) - x(0)) + 3X(s) = \frac{5}{s^2+25}\,.$$

Substituting the initial conditions into this equation we have after some simplification:

$$\left(2s^2 - s + 3\right)X(s) - 2s + 5 = \frac{5}{s^2+25}\,.$$

Solving for $X(s)$ we have

$$X(s) = \frac{2s-5}{2s^2-s+3} + \frac{5}{(s^2+25)(2s^2-s+3)}\,.$$

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4. (20) At least one of the following differential operators in non-linear and at least one is linear. Find one which is non-linear and show that it is non-linear, and then find one which is linear and show that it is linear.

(a) $L[u] = \dfrac{d^2u}{dt^2} + u\sin(t^2)$.

Ans: This operator is linear.

$$\begin{aligned} L[f+g] &= (f+g)'' + \sin(t^2)(f+g) \\ &= f'' + g'' + \sin(t^2)f + \sin(t^2)g \\ &= L[f] + l[g]\,. \end{aligned}$$

$$\begin{aligned} L[\alpha f] &= (\alpha f)'' + \sin(t^2)(\alpha f) \\ &= \alpha f'' + \alpha(\sin(t^2)f) \\ &= \alpha L[f]\,. \end{aligned}$$

(b) $L[u] = t^2\dfrac{d^2u}{dt^2} + \dfrac{du}{dt} + 2u$.

Ans: This operator is also linear. Check as in the previous question.

(c) $L[u] = t\dfrac{d^2u}{dt^2} + u\dfrac{du}{dt}$.

Ans: This operator is nonlinear. The scalar multiplication requirement is seen to be invalid for all values of α.

$$\begin{aligned} L[\alpha f] &= t(\alpha u)'' + (\alpha u)(\alpha u)' \\ &= \alpha t u'' + \alpha^2 u u'\,. \end{aligned}$$

This last term is equal to $\alpha L[u]$ only when $\alpha = \alpha^2$.

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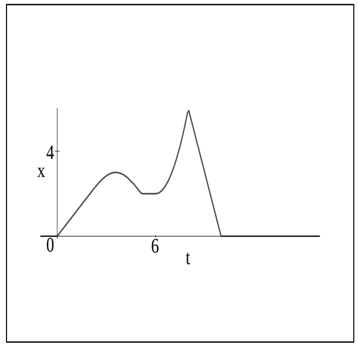
5. (20) Can a function whose graph looks like the following plot be a solution to a differential equation of the form $ax'' + bx' + cx = 0$, where a, b, and c are positive constants. Be sure to explain why or why not such a function can be a solution to this differential equation.



Ans: The function whose plot is given above cannot be a solution to the given differential equation. All solutions of equations of this type involve exponentials or exponentials times sines or cosines. Such functions are not linear on any interval and most certainly are not equal to zero on an interval and then not zero on some other interval.

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6. (30) Find all of the equilibrium points of the system below, find the linear approximation of the system about each of the equilibrium points, and determine, if possible, what the linear approximations predict about solutions of the original system.

$$\frac{dx}{dt} = xy + 2x^2$$
$$\frac{dy}{dt} = -x + xy + y^2$$

Ans: The equilibrium points are $(0,0)$ and $(1/2,-1)$. The Jacobian matrix of this system equals

$$Jf(x,y) = \begin{bmatrix} y+4x & x \\ -1+y & x+2y \end{bmatrix}.$$

At the equilibrium point $(0,0)$ Jf equals $\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$. Zero is the only eigenvalue of this matrix. Since the eigenvalues do not have a positive or negative real part the linearized system cannot tell us anything about the behavior of solutions of the original system that are close to the point $(0,0)$.

At the equilibrium point $(1/2,-1)$, the Jacobian equals

$$\begin{bmatrix} 1 & 1/2 \\ -2 & -3/2 \end{bmatrix}.$$

The eigenvalues of this matrix are -1 and $1/2$. The associated eigenvectors are $[1,-4]$ and $[-1,1]$ respectively. Thus, since both eigenvalues have non-zero real part, the linear system does tell us the behavior of solutions to the non-linear system. This equilibrium point is a saddle point. There is a curve passing through the point $(1/2,-1)$ which is tangent to the vector $[1,-4]$ and solutions which start on this curve stay on the curve and tend to the equilibrium point. There is a second curve through this point which is tangent to the vector $[-1,1]$ and solutions which start on this curve stay on the curve and move away from the equilibrium point. In general any solution which starts close to this point, unless it lies on the first curve, will move away from the point.

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7. (20) A geneticist has a colony of rats whose population she models with a logistic differential equation. If the initial population of the rats is 5, the population at t=1 is 10, and the population at t=2 is 15, what is the expected limiting value of the population as $t \to \infty$?

Ans: The rat population is modeled by a differential equation of the form

$$\frac{dP}{dt} = kP(m - P)\,,$$

with the initial condition $P(0) = 5$. The general solution to this problem is $P(t) = \dfrac{5m}{5 + exp(-kmt)(m-5)}$. Using the other two bits of data we derive two equations and get Maple to solve them for k and m. This gives us the values $k = \ln 3/20$ and $m = 20$. Thus, the solution to this logistic equation which satisfies the given data is

$$P(t) = \frac{20}{1 + 3e^{-t\ln 3}}\,.$$

The limiting value of this function as $t \to \infty$ is $m = 20$.

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8. (20) A mass spring system is being designed and the damping coefficient has not been determined. The mass is 2 and the spring constant is 3. How large must the damping coefficient be in order for the system to be critically damped. Find the general solution to this equation in that case.

Ans: A differential equation which models this system is

$$2x'' + k_d x' + 3x = 0\,.$$

The roots of the associated algebraic equation are $r = \left(-k_d \pm \sqrt{k_d^2 - 24}\,\right)/4$. Thus, the system is critically damped if $k_d = \sqrt{24}$. The general solution to this system in this case is

$$x(t) = c_1 e^{-\frac{\sqrt{24}}{4}t} + c_2 t e^{-\frac{\sqrt{24}}{4}t}\,.$$

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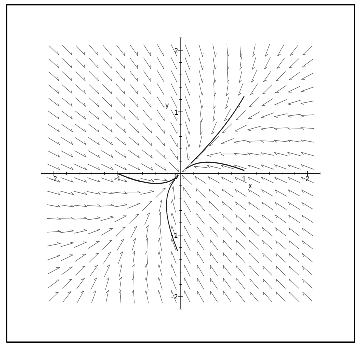
9. (30) Each of the following sets of eigenvalues and eigenvectors arise from the coefficient matrix of a two dimensional linear system of first order differential equations. Sketch the phase plane for each system, be sure to indicate the asymptotic behavior of solutions.

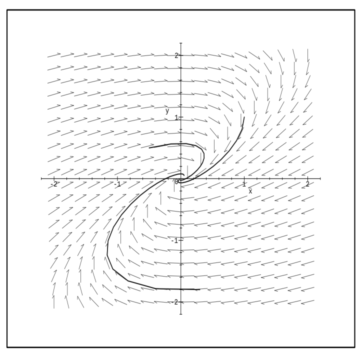
(a) $\lambda_1 = -2\,, \quad \vec{\xi_1} = (1,1)\,, \quad \lambda_2 = -5\,, \vec{\xi_2} = (-1,1)$.

Ans: Since both eigenvalues are negative real numbers there are two straight line solutions. The one corresponding to the eigenvalue -5 is given by the vector $(-1,1)$ and the one corresponding to the eigenvalue -2 is given by the vector $(1,1)$. The second line gives the asymptotic direction of all solutions as $t \to \infty$.



(b) $\lambda_1 = -1+i\,, \quad \vec{\xi_1} = (1+i, i)$.

Ans: Here the eigenvalues are complex with negative real part. Thus, the origin is a spiral sink.



(c) What is the general real valued vector solution to the system of part (b)?

Ans: One complex vector valued solution is

$$X(t) = e^{(-1+i)t} \begin{bmatrix} 1+i \\ i \end{bmatrix} = e^{-t} (\cos(t) + i \sin(t)) \begin{bmatrix} 1+i \\ i \end{bmatrix}$$
$$= e^{-t} \left\{ \begin{bmatrix} \cos t - \sin t \\ -\sin t \end{bmatrix} + i \begin{bmatrix} \cos t + \sin t \\ \cos t \end{bmatrix} \right\}.$$

Taking the real and imaginary parts of this solution we can construct the general real valued solution, which is

$$X(t) = e^{-t} \left\{ c_1 \begin{bmatrix} \cos t - \sin t \\ -\sin t \end{bmatrix} + c_2 \begin{bmatrix} \cos t + \sin t \\ \cos t \end{bmatrix} \right\}.$$
