1. (10) Solve the following initial value problem:

$$\frac{dx}{dt} = tx, \quad x(0) = 2$$

The equation is separable and can be solved as follows

$$\begin{aligned} \frac{x'}{x} &= t \\ \ln|x| &= \frac{t^2}{2} + c \\ |x| &= ce^{t^2/2} \quad \text{different } c \\ x(t) &= 2e^{t^2/2} \end{aligned}$$

2. (10) Sketch the phase line for the differential equation

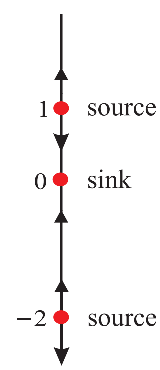
$$\frac{dx}{dt} = x^3 + x^2 - 2x\,.$$

Be sure to determine the type of each equilibrium point.

The equilibrium points for this differential equation are found by setting $f(x) = x^3 + x^2 - 2x = x(x+2)(x-1) = 0$ and solving for x. These values are 0, -2, and 1. The partial derivative of $f(x)$ is evaluated at each of the equilibrium points to try to determine their type.

$$\begin{aligned} \frac{\partial f}{\partial x} &= 3x^2 + 2x - 2 \\ \left.\frac{\partial f}{\partial x}\right|_{x=-2} &= 6, \quad x = -2 \text{ is a source.} \\ \left.\frac{\partial f}{\partial x}\right|_{x=0} &= -2, \quad x = 0 \text{ is a sink} \\ \left.\frac{\partial f}{\partial x}\right|_{x=1} &= 3, \quad x = 0 \text{ is a source} \end{aligned}$$

The phase line is shown below.



3. (20) A tank holds 50 gallons of pure water. Suppose 5 gallons of water which contains 5/2 pounds of salt per gallon is added to the tank every minute, and at the same time 5 gallons of the solution in the tank is taken out every minute. Assume that the salt being added to the tank is immediately uniformly distributed throughout the tank. Find a formula which gives the number of pounds of salt in the tank at any minute. Then use your formula to determine the limiting amount of salt in the tank as t tends to infinity.

   Let $S(t)$ denote the number of pounds of salt in the tank at time t in minutes. Then $S(t)$ satisfies the following initial value problem

$$\frac{dS}{dt} = \frac{5}{2}(5) - \frac{S}{50}(5), \quad S(0) = 0$$

   The solution to this first order linear initial value problem is

$$S(t) = 125\left(1 - e^{-t/10}\right)$$

   The limit of this expression as t goes to infinity is 125, which agrees with our intuition. The concentration of sugar as t goes to infinity should be 5/2 pounds per gallon. There are 50 gallons, so we should wind up with $50(5/2)$ pounds of sugar.

4. (20) Suppose that -2 and -5 are eigenvalues with eigenvectors $(1, 2)$ and $(-3, 1)$ respectively of the coefficient matrix for the system of differential equations

$$\vec{X}\,' = A\vec{X},$$

   where $\vec{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

   (a) Find the solution to the initial value problem $\vec{X}' = A\vec{X}, \quad \vec{X}(0) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

   The general solution to this system is

$$\vec{X}(t) = c_1 e^{-2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-5t}\begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

   The initial condition implies that the constants c_1 and c_2 satisfy the following equation

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

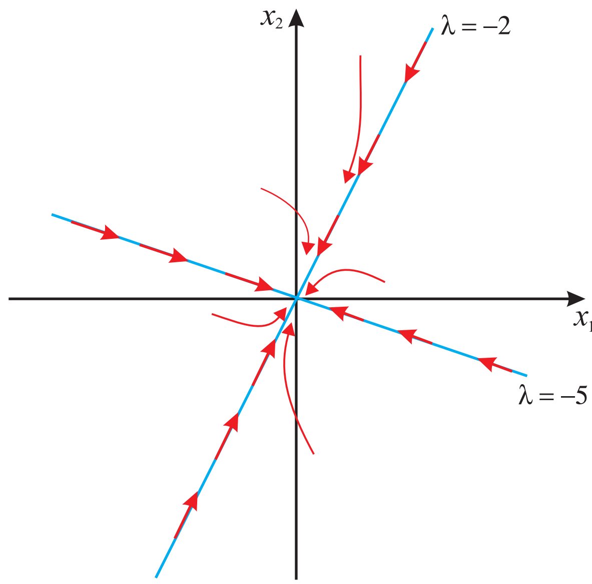
   or

$$\begin{aligned} c_1 - 3c_2 &= 3 \\ 2c_1 + 4c_2 &= 4 \end{aligned}$$

   The solution to this system of equations is $c_1 = 15/7$ and $c_2 = -2/7$. Thus, the solution to the initial value problem is

$$\vec{X}(t) = \frac{15}{7}e^{-2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{2}{7}e^{-5t}\begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

(b) Sketch the direction field for this system of differential equations. Pay particular attention to the asymptotic behavior of solutions as t tends to infinity. Be sure to sketch the "straight line" solutions.



5. (20) The following questions refer to the second order differential equation

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 0\,.$$

(a) Write this differential equation as a system of differential equations.

Set $x_1 = y$ and $x_2 = y'$. Then x_1 and x_2 satisfy the system

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= -x_1 - x_2 \end{aligned}$$

In vector notation, setting $\vec{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ we have

$$\frac{d\vec{X}}{dt} = \begin{bmatrix} x_2 \\ -x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(b) Find the eigenvalues of the coefficient matrix of this system. You do not need to find the eigenvectors.

The eigenvalues of the coefficient matrix are the roots of the equation

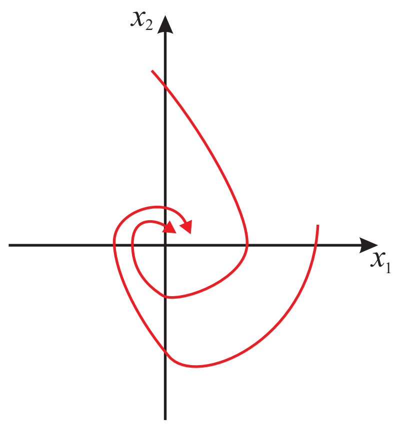
$$\begin{aligned} \det\left(\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) &= 0 \\ \det\left(\begin{bmatrix} -\lambda & 1 \\ -1 & -1-\lambda \end{bmatrix} \right) &= 0 \\ \lambda^2 + \lambda + 1 &= 0 \end{aligned}$$

and they equal

$$\lambda = \frac{-1 \pm \sqrt{3}i}{2}$$

Note that the eigenvalues are complex with negative real part.

(c) Sketch the phase plane and direction field for this system.



The origin is a spiral sink.

(d) From your knowledge of the eigenvalues of this system describe the behavior of a solution to the original second order differential equation.

Since the eigenvalues are complex with negative real part. A solution $y(t)$ of the original second order differential equation will oscillate about the value 0, and as t becomes infinite the solutions will converge to 0.

6. (10) Let $y(t)$ be a solution of the initial value problem $\frac{dy}{dt} = t - y^2 + ty$, $y(1) = 2$

Determine the values of $y'(1)$ and $y''(1)$.

$$\left.\frac{dy}{dt}\right|_{t=1} = 1 - 4 + 2 = -1$$

$$\begin{aligned} \left.\frac{d^2y}{dt^2}\right|_{t=1} &= \left.(1 - 2yy' + y + ty')\right|_{t=1} = 1 - 2(2)(-1) + 2 + (1)(-1) \\ &= 6 \end{aligned}$$

7. (10) Explain how you would use Euler's method to estimate the solution of the following initial value problem at $t = 4$.

$$\begin{aligned} \frac{dx_1}{dt} &= tx_1x_2 - x_1\,, \quad x_1(0) = 2 \\ \frac{dx_2}{dt} &= x_2^2 - x_1x_2\,, \quad x_2(0) = 3 \end{aligned}$$

Let $f_1(x_1,\, x_2,\, t) = tx_1x_2 - x_1$ and $f_2(x_1,\, x_2,\, t) = x_2^2 - x_1x_2$. To approximate the solution at $t = 4$, I would decide on how many subintervals to divide $[0, 4]$ into. Call this number N, and set $h = 4/N$. Denote the approximation of x_1 and x_2 at the $\mathrm{i^{th}}$ step by $x_{1,i}$ and $x_{2,i}$ respectively for $i = 0,\, 1,\, \cdots,\, N$, where $x_{1,0} = x_1(0) = 2$ and $x_{2,0} = x_2(0) = 3$. Set $t_i = 0 + ih$ for $i = 0,\, 1,\, \cdots,\, N$. The iteration scheme is as follows

$$\begin{aligned} x_{1,i+1} &= x_{1,i} + hf_1(x_{1,i}, x_{2,i}, t_i) \\ x_{2,i+1} &= x_{2,i} + hf_2(x_{1,i}, x_{2,i}, t_i) \end{aligned}$$

for $i = 0,\, \cdots,\, N - 1$. The values $x_{1,N}$ and $x_{2,N}$ are approximations to $x_1(4)$ and $x_2(4)$ respectively.