1. (10) For the system of differential equations below: what is the differential operator associated with this system, and is the system linear or non-linear?

$$\frac{dx}{dt} = ty - x\cos t + t^2$$
$$\frac{dy}{dt} = x - y - \sin t\,.$$

The differential operator is

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} ty - x\cos t \\ x - y \end{bmatrix},$$

and it is a linear operator, which means that it is a linear system. The proof of the operator's linearity follows.

$$\begin{aligned}
L\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) &= L\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}\right) \\
&= \frac{d}{dt}\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} - \begin{bmatrix} t(y_1 + y_2) - (x_1 + x_2)\cos t \\ (x_1 + x_2) - (y_1 + y_2) \end{bmatrix} \\
&= \frac{d}{dt}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \frac{d}{dt}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} - \begin{bmatrix} ty_1 - x_1\cos t \\ x_1 - y_1 \end{bmatrix} - \begin{bmatrix} ty_2 - x_2\cos t \\ x_2 - y_2 \end{bmatrix} \\
&= L\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right),
\end{aligned}$$

and

$$\begin{aligned}
L\left(\alpha\begin{bmatrix} x \\ y \end{bmatrix}\right) &= L\left(\begin{bmatrix} \alpha x \\ \alpha y \end{bmatrix}\right) \\
&= \frac{d}{dt}\begin{bmatrix} \alpha x \\ \alpha y \end{bmatrix} - \begin{bmatrix} t(\alpha y) - (\alpha x)\cos t \\ (\alpha x) - \alpha y \end{bmatrix} \\
&= \alpha\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} - \alpha\begin{bmatrix} ty - x\cos t \\ x - y \end{bmatrix} \\
&= \alpha L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)
\end{aligned}$$

2. (15) Find a particular solution of the system

$$\frac{d(\vec{x})}{dt} = \begin{bmatrix} 4 & 10 \\ 5 & -1 \end{bmatrix} \vec{x} + \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

Since the non-homogeneous term is a constant we look for a constant particular solution, $\vec{x}_p$. This means that $\vec{x}_p$ must satisfy the equation

$$\begin{bmatrix} 4 & 10 \\ 5 & -1 \end{bmatrix} \vec{x}_p + \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} 4 & 10 \\ 5 & -1 \end{bmatrix} \vec{x}_p = -\begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

The solution to this equation is

$$\vec{x}_p = \begin{bmatrix} -\frac{11}{18} \\ -\frac{1}{18} \end{bmatrix}.$$

3. (40) Suppose the 2×2 matrix A of the following initial value problem

$$\vec{x}' = A\vec{x}$$
$$\vec{x}(0) = \begin{bmatrix} -2 \\ 3 \end{bmatrix},$$

has the following eigenvalues and eigenvectors:

λ	-1	2
$\vec{x}_\lambda$	$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

a. Find a fundamental solution set for this system. Be sure to compute the Wronskian of the functions you say form a fundamental solution set.

Two solutions to this homogeneous system are

$$\vec{\phi}_1(t) = e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$\vec{\phi}_2(t) = e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

They form a fundamental solution set if their Wronskian is non-zero; the value of the Wronskian is

$$W\left(\vec{\phi}_1, \vec{\phi}_2\right) = \det\begin{bmatrix} e^{-t} & e^{2t} \\ -e^{-t} & e^{2t} \end{bmatrix} = 2e^t \neq 0\,.$$

b. Explain why the functions you selected in part a. are solutions to this system.

These two functions are solutions because of the following. Let r be an eigenvalue of the matrix A with $\vec{x}_r$ a corresponding eigenvector. Then the function $\vec{\phi}(t) = e^{rt}\,\vec{x}_r$ satisfies

$$\frac{d\vec{\phi}}{dt} = re^{rt}\,\vec{x}_r$$
$$A\vec{\phi} = A\left(e^{rt}\,\vec{x}_r\right) = e^{rt}\,(A\vec{x}_r) = e^{rt}\,(r\vec{x}_r)\,.$$

Thus, the function $\vec{\phi}$ satisfies the differential equation $\vec{x}' = A\vec{x}'$

c. Find the solution to the initial value problem.

The general solution to this system of differential equations is

$$\vec{x} = c_1e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

To satisfy the initial conditions the constants c_1 and c_2 must be such that

$$\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \vec{x}(0) = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$c_1 + c_2 = -2, \quad -c_1 + c_2 = 3$$
$$c_1 = -\frac{5}{2}, \quad c_2 = \frac{1}{2}\,.$$

Thus, the solution to the initial value problem is

$$\vec{x} = -\frac{5e^{-t}}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{e^{2t}}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

4. (35) For the nonlinear system shown below

$$\frac{dx}{dt} = (x+y)(x-2)$$
$$\frac{dy}{dt} = x(y-3)$$

a. What arrows would you draw in the phase plane for this system at the points $(-2,3)$ and $(4,2)$?

Let $F(x,y) = \begin{bmatrix} (x+y)(x-2) \\ x(y-3) \end{bmatrix}$. Then, the arrow that would be drawn at any point (x_0, y_0) is $F(x_0, y_0)$, or a scaled version of this vector. At the points in question we have

$$(-2,3): F(-2,3) = \begin{bmatrix} (-2+3)((-2)-2) \\ (-2)(3-3) \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

$$(4,2): F(4,2) = \begin{bmatrix} (4+2)(4-2) \\ 4(2-3) \end{bmatrix} = \begin{bmatrix} 12 \\ -4 \end{bmatrix}$$

b. Find all equilibrium points.

An equilibrium point is a point at which the function F defined in part a is $\vec{0}$. That is, we're looking for points (x,y) such that

$$(x+y)(x-2) = 0$$
$$x(y-3) = 0\,.$$

There are several cases: first suppose $x = 2$, then $y = 3$, then if $x = 0$, we must have $y = 0$. The last case is $x \neq 0$, which forces $y = 3$. Since $x = 2$ is already covered all that is left is for $x = y = 0$ or $x = -y = -3$. Thus, there are three equilibrium points:

$$(0,0),\ (2,3),\text{ and } (-3,3)\,.$$

c. If (x_0, y_0) is an equilibrium point, find the linear approximation to the non-linear system about this point.

The linear approximation to this system is

$$\frac{d}{dt}(\vec{u}) = \left.\begin{bmatrix} \frac{d}{dx}[(x+y)(x-2)] & \frac{d}{dy}[(x+y)(x-2)] \\ \frac{d}{dx}[x(y-3)] & \frac{d}{dy}[x(y-3)] \end{bmatrix}\right|_{(x_0,y_0)} \vec{u}$$

$$= \left.\begin{bmatrix} 2x+y-2 & x-2 \\ y-3 & x \end{bmatrix}\right|_{(x_0,y_0)} \vec{u}$$

$$\begin{bmatrix} 2x_0+y_0-2 & x_0-2 \\ y_0-3 & x_0 \end{bmatrix} \vec{u}\,.$$

d. One of the points you should have found in part a. is $(-3,3)$. Using the linear approximation to this system at this point describe the behavior of solutions to the non-linear system.

At the point $(-3,3)$. the matrix A of the linear approximation is

$$A = \left.\begin{bmatrix} 2x_0+y_0-2 & x_0-2 \\ y_0-3 & x_0 \end{bmatrix}\right|_{(-3,3)} = \begin{bmatrix} -5 & -5 \\ 0 & -3 \end{bmatrix}.$$

The eigenvalues of this matrix are -5 and -3. Since both of them are negative, all solutions to the non-linear system that start close enough to the point $(-3,3)$ will converge to this point as t goes to infinity.