1. (10) What is the general solution to the differential equation $x' = 2x$. No Maple on this one.

Ans: $x(t) = ce^{2t}$.

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2. (20) Determine whether each of the following equations or systems of equations is linear. **Be sure to explain** your answer using the definition of linearity.

(a) $\dfrac{dy}{dx} = x^2 - 3y + 1$.

Ans: The differential operator associated with this equation is $L[y] = y' + 3y$. An easy check shows that L satisfies to two conditions of linearity. That is,

$$L[y_1 + y_2] = L[y_1] + L[y_2]$$
$$L[\alpha y] = \alpha L[y]$$

Thus, this is a linear equation.

(b) $\dfrac{dx}{dt} = x - 3y + 1$, $\dfrac{dy}{dt} = 7x + 4y$.

Ans: The differential operator for this system is:

$$L\left[\begin{bmatrix} x \\ y \end{bmatrix}\right] = \begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} x - 3y \\ 7x + 4y \end{bmatrix}.$$

It is easy to verify that this operator is linear. Therefore this system is also linear.

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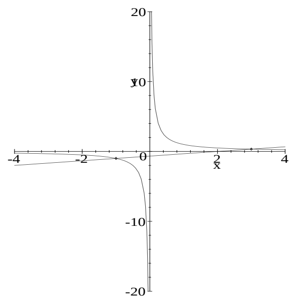
3. (45) Consider the following non-linear system:

$$x' = 1 - xy$$
$$y' = x - 3y - 2$$

(a) Sketch the null-clines for this system, locate all equilibrium points, and indicate the direction of motion of solution curves as they cross the null-clines.

Ans: There are two equilibrium points, $(-1, 1)$, and $(3, 1/3)$.



(b) Find the linear approximation to this system about each of the equilibrium points. From your knowledge of the linear approximation, what type is the equilibrium point?

Ans: The Jacobean matrix equals $\begin{bmatrix} -y & -x \\ 1 & -3 \end{bmatrix}$. At the equilibrium point $(-1, -1)$, the linear approximation is:

$$u' = u + v$$
$$v' = u - 3v\,.$$

The eigenvalues of the coefficient matrix are $-1 \pm \sqrt{5}$. Thus, the equilibrium point $(-1, -1)$ is a saddle point.

At the equilibrium point $(3, 1/3)$ the linear approximation is

$$u' = -\frac{u}{3} - 3v$$
$$v' = u - 3v\,.$$

The eigenvalues of this system are $\dfrac{-5 \pm \sqrt{11}\,i}{3}$. Thus, the equilibrium point $(3, 1/3)$ is a spiral sink.

(c) Is this a Hamiltonian system? Why or why not?

Ans: The system is not Hamiltonian. If it was Hamiltonian, then the following equation would have to be valid:

$$\frac{\partial}{\partial x}(1 - xy) = \frac{\partial}{\partial y}(x - 3y - 2)\,.$$

An easy computation shows that the equation is not true.

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4. (30) Suppose the function $y(t)$ satisfies the initial value problem:

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y(t) = \sin(2t)\,, \quad y(0) = -1\,, \quad y'(0) = 4\,.$$

(a) If $Y(s)$ denotes the Laplace transform of the solution, $y(t)$, what equation does $Y(s)$ satisfy?

Ans: The function $Y(s)$ satisfies the equation

$$s(sY(s) + 1) - 5 - sY(s) - 2Y(s) = \frac{2}{s^2 + 4}$$

or

$$Y(s) = \frac{13}{12}\frac{1}{s-2} - \frac{32}{15}\frac{1}{s+1} + \frac{1}{20}\frac{s-6}{s^2+4}$$

(b) Find the inverse Laplace transform of $\dfrac{1}{s^2 - s - 2}$.

Ans: $\mathcal{L}^{-1}\left(\dfrac{1}{s^2 - s - 2}\right) = \dfrac{1}{3}e^{2t} - \dfrac{1}{3}e^{-t}.$

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5. (30) Suppose you drop a ball straight down from a height of 5 meters, and as soon as the ball hits the ground it receives an impulse force of 2 Newtons which is directed vertically upward. Assume the mass of the ball is 0.2 kilograms, that air resistance exerts a force oppositely directed to the velocity of the ball, and that the magnitude of this force is .02 times the magnitude of the velocity of the ball. If g denotes the acceleration due to gravity, then $g = 9.8$ meters per second per second.

(a) Model this problem with a differential equation which uses the Dirac-delta function.

Ans: An equation which models this is

$$m\frac{d^2y}{dt^2} = -9.8m - 0.02\frac{dy}{dt} + 2\delta(t - t_1)\,,\ t_1 \approx 1.02745 \text{ seconds}$$

where $m = 0.2$ kilograms, and t_1 is the time the ball first hits the ground. .

(b) How high will the ball go after the impulse force is applied.

Ans: Solve the differential equation, then set its derivative equal to zero. The solution to this equation for $t > 0$ is $t_{max} \approx 1.0713$ seconds. The height of the ball at that time is the maximum height attained after the impulse force is applied and equals $y_{max-height} \approx 0.009$ meters.

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6. (30) Consider the initial value problem:

$$\frac{d^2y}{dt^2} + 4y = \sin \omega t\,,\ y(0) = 0\,,\ y'(0) = 0\,.$$

(a) Find the solution to this initial value problem.

Ans: The solution if $\omega \neq \pm 2$ is

$$y(t) = \frac{\omega \sin t \cos t}{(\omega - 2)(\omega + 2)} - \frac{\sin \omega t}{(\omega - 2)(\omega + 2)}\,.$$

If $\omega = 2$, the solution is

$$y(t) = \frac{\sin 2t}{8} - \frac{t}{4}\cos 2t\,.$$

Since sin is an odd function, the negative of this solution works for the case $\omega = -2$.

(b) Explain resonance.

Ans: Resonance occurs when a differential equation has a periodic forcing function and the period of this forcing function is the same (or almost the same) as the period of solutions to the homogeneous equation associated with the given non-homogeneous equation. The effect of resonance is unusually large amplitudes which can become unbounded as time becomes arbitrarily large.

(c) For what values of ω will resonance occur in this problem.

Ans: $\omega = 2$.
