Print your name and course number (**LEGIBLY**) on the front cover of your bluebook. If you do not have a blue book, you may work the exam on 8.5 by 11 sheets of paper. Be sure to staple or paper clip sheets of paper together before turning them in. Show all work **neatly** in numerical order. You do not need to return this exam sheet.

Note that explanations are required. An answer with no explanation is worth zero points.

1. (25) Describe the solution set of the following system of equations:

$$\begin{aligned} 2x_1 + 5x_2 - 5x_3 + 21x_4 &= 6 \\ x_1 + 2x_2 - 2x_3 + 8x_4 &= 3 \\ 15x_1 + 18x_2 - 19x_3 + 57x_4 &= 37 \end{aligned}$$

Ans: The augmented matrix of this system $\begin{bmatrix} 2 & 5 & -5 & 21 & 6 \\ 1 & 2 & -2 & 8 & 3 \\ 15 & 18 & -19 & 57 & 37 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & 0 & 0 & -2 & 3 \\ 0 & 1 & 0 & 8 & 8 \\ 0 & 0 & 1 & 3 & 8 \end{bmatrix}$. Thus, the solution set is the set of points (x_1, x_2, x_3, x_4), where $x_1 = 2x_4 + 3$, $x_2 = -8x_4 + 8$, and $x_3 = -3x_4 + 8$. The variable x_4 can be any number.

---

2. (20) Let V be the vector space of all 3×2 matrices with real entries. Let W be that subset of V which consists of all matrices for which the entries in each column sum to zero. That is, if A is in W and $A = [a_{i,j}]$, then for each j, we have $\sum_{i=1}^{3} a_{i,j} = 0$.

   Let $B = \left\{ \begin{bmatrix} 1 & 2 \\ -3 & 1 \\ 2 & -3 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -3 & 0 \\ 2 & 0 \end{bmatrix} \right\}$

   (a) Is W a subspace of V?

Ans: Yes, W is a subspace. To verify this, we need to check that W is nonempty, which it is since the the zero vector of V is in W. Now we only need to verify that W is closed under scalar multiplication and vector addition. So let A be an arbitrary vector in W, and let α be an arbitrary scalar. Then $\alpha A = \alpha[a_{i,j}] = [\alpha a_{i,j}]$, and for each j we have $\sum_{i=1}^{3} \alpha a_{i,j} = \alpha \sum_{i=1}^{3} a_{i,j} = \alpha 0 = 0$. Thus, W is closed under scalar multiplication. To verify that W is closed under vector addition let A and B be two arbitrary vectors in W. Then $A + B = [a_{i,j}] + [b_{i,j}] = [a_{i,j} + b_{i,j}]$, and for each j we have $\sum_{i=1}^{3}(a_{i,j}+b_{i,j}) = \sum_{i=1}^{3} a_{i,j} + \sum_{i=1}^{3} b_{i,j} = 0+0=0$. Thus, W is also closed under vector addition, and W is a subspace.

(b) Is the set B linearly independent?

Ans: Yes, the set B is linearly independent. To see this suppose we have constants a, b, c, and d such that $a\begin{bmatrix} 1 & 2 \\ -3 & 1 \\ 2 & -3 \end{bmatrix} + b\begin{bmatrix} 0 & -1 \\ 1 & 1 \\ -1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} +$ $d\begin{bmatrix} 1 & 0 \\ -3 & 0 \\ 2 & 0 \end{bmatrix} = \vec{0}$, where $\vec{0}$ is the 3×2 zero matrix. This matrix equation leads to a homogeneous system of 6 equations in 4 unknowns. The coefficient matrix of this system is the first matrix below

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & -1 & 0 & 0 \\ -3 & 1 & 0 & -3 \\ 1 & 1 & 1 & 0 \\ 2 & -1 & 0 & 2 \\ -3 & 0 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and it is row equivalent to the second matrix above. From the row reduced echelon form above we see that $a=b=c=d=0$. Thus, the given set of vectors is linearly independent.

---

3. (20) Let $W_0 = \{\,(x_1,x_2,x_3,x_4)\ :\ x_1+3x_2-4x_3=0,\ x_2+2x_3-x_4=0\,\}$.
Let $W_1 = \{\,(x_1,x_2,x_3,x_4)\ :\ x_1+3x_2-4x_3=1,\ x_2+2x_3-x_4=1\,\}$.

(a) One of the subsets W_0 or W_1 of R^4 is a subspace of R^4. Which one is a subspace and why, and why isn't the other one a subspace.

Ans: The set W_0 is a subspace and W_1 is not. To see that W_1 is not a subspace just notice that the zero vector is not in W_1. It is an easy matter to show that W_0 is a subspace. First note that $\vec{0}\in W_0$. Then just show that W_0 is closed under scalar multiplication and vector addition.

(b) Find a basis for your answer to part a.

Ans: The subspace W_0 is the solution set for a system of homogeneous equations. The coefficient matrix of the system and its row reduced echelon form are:

$$\begin{bmatrix} 1 & 3 & -4 & 0 \\ 0 & 1 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -10 & 3 \\ 0 & 1 & 2 & -1 \end{bmatrix}.$$

Thus, we have $x_1 = 10x_3 - 3x_4$, and $x_2 = -2x_3 + x_4$. Setting the free variables equal to 1 and 0 and then equal to 0 and 1, we get the following two vectors: $(10, -2, 1, 0)$ and $(-3, 1, 0, 1)$, respectively. Clearly these two vectors are linearly independent (look at the last two slots). To see that they span W_0 Let $\vec{x}$ be any vector in W_0. Then $\vec{x} = (10x_3 - 3x_4, -2x_3 + x_4, x_3, x_4) = x_3(10, -2, 1, 0) + x_4(-3, 1, 0, 1)$. Thus, these two vectors are a linearly independent spanning set of W_0. That is, a basis.

---

4. (20) Define the following "inner product" on R^2:

$$< (x_1, x_2), (y_1, y_2) > = 2x_1y_1 - 2(x_1y_2 + x_2y_1) + 3x_2y_2 \,.$$

(a) Show that this "product" satisfies the three requirements of an inner product.

Ans: To see that this product is positive definite we compute $< \vec{x}, \vec{x} >$.

$$\begin{aligned} < \vec{x}, \vec{x} > &= 2x_1^2 - 4x_1x_2 + 3x_2^2 = 2x_1^2 - 4x_1x_2 + 2x_2^2 + x_2^2 \\ &= \left(\sqrt{2}x_1 - \sqrt{2}x_2\right)^2 + x_2^2 \end{aligned}$$

Clearly this expression is never negative, and if it equals zero we must have $x_1 = x_2 = 0$. It is obvious that $< \vec{x}, \vec{y} > = < \vec{y}, \vec{x} >$. All that is left to verify is that this product is linear in the first slot.

$$\begin{aligned} < \alpha x \vec{+} \beta y, \vec{z} > &= 2(\alpha x_1 + \beta y_1)z_1 - 2[(\alpha x_1 + \beta y_1)z_2 + (\alpha x_2 + \beta y_2)z_1] \\ &\quad +3(\alpha x_2 + \beta y_2)z_2 \\ &= \alpha[2x_1z_1 - 2(x_1z_2 + x_2z_1) + 3x_2z_2] \\ &\quad +\beta[2y_1z_1 - 2(y_1z_2 + y_2z_1) + 3y_2z_2] \\ &= \alpha < \vec{x}, \vec{z} > +\beta < \vec{y}, \vec{z} > \end{aligned}$$

(b) Let $\vec{x} = (1, 0)$ and $\vec{y} = (0, 1)$. Using this inner product find the length of $\vec{x}$, and the angle between the two vectors $\vec{x}$ and $\vec{y}$.

Ans: $||\vec{x}||^2 = < \vec{x}, \vec{x} > = < (1\vec{,}0), (1\vec{,}0) > = 2$. Thus, the length of $\vec{x}$ is $\sqrt{2}$. To compute the angle between the two vectors we use the formula:

$$\cos(\theta) = \frac{< \vec{x}, \vec{y} >}{||\vec{x}||\,||\vec{y}||} = \frac{-2}{\sqrt{2}\sqrt{3}} \approx -0.8165\,.$$

Thus, the angle between the two vectors (in degrees) is $\approx \arccos(-0.8165) \approx 144.74°$.

---

5. (15) Define each of the following terms and give an example of each term. That is, if I asked you to define a vector space and provide an example, your answer would be the definition and an explicit vector space, e.g. R^2.

   (a) linearly independent set.

Ans: A set $\{ x_1, \cdots, x_k \}$ is linearly independent if whenever we have a linear combination of these vectors equal to the zero vector, then it is the trivial linear combination. That is, if

$$\sum_{i=1}^{k} c_i \vec{x}_i = \vec{0}\,,$$

then $c_i = 0$ for each i. An example of a linearly independent set of vectors is $\{ (1,0),\ (0,1) \}$.

   (b) homogeneous system of equations.

Ans: A system of equations is said to be homogeneous if every equation in the system can be written in the form $a_{i,1}x_1 + \cdots + a_{i,n}x_n = 0$. That is the left hand side of the equation is a linear combination of the unknowns x_i, and the right hand side is zero. An example of a homogeneous system is:

$$\begin{aligned} 2x_1 - 3x_2 &= 0 \\ 5x_1 + x_2 &= 0\,. \end{aligned}$$

   (c) Elementary row matrix.

Ans: An elementary row matrix is any matrix which can be obtained by performing an elemtary row operation on an identity matrix. An example is:

$$E = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

This is the elementary row matrix that is associated with the elementary row operation $2R_2 + R_1$.
