Print your name and course number (**LEGIBLY**) on the front cover of your bluebook. If you do not have a blue book, you may work the exam on 8.5 by 11 sheets of paper. Be sure to staple or paper clip sheets of paper together before turning them in. Show all work **neatly** in numerical order. You do not need to return this exam sheet.

Note that explanations are required. An answer with no explanation is worth zero points.

1. (15) Suppose that $\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$ are three linearly independent vectors in some vector space. Explain what the Gram-Schmidt algorithm is, and how it would be used to construct an orthogonal basis for the subspace $W = \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$

Ans: The Gram-Schmidt algorithm is a technique to construct an orthogonal set of vectors from a given set of linearly independent vectors. For three vectors the algorithm is as follows:

$$\begin{aligned}\vec{v}_1 &= \vec{x}_1\\ \vec{v}_2 &= \vec{x}_2 - PROJ_{\vec{v}_1}\vec{x}_2\\ \vec{v}_3 &= \vec{x}_3 - (PROJ_{\vec{v}_1}\vec{x}_3 + PROJ_{\vec{v}_2}\vec{x}_3)\end{aligned}$$

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2. (10) Let $W = \text{span}\{(1,2,4,5), (1,-1,2,3)\}$. Let $\vec{b} = (1,-1,3,-4)$. Find the projection of $\vec{b}$ onto W, and the distance from $\vec{b}$ to W. Use the standard inner product in R^4.

Ans: Using the Gram-Schmidt procedure an orthogonal basis for W is constructed. It is $\{[1,2,4,5],\ [0.522,-1.957,0.087,0.609]\}$. This basis is then used to calculate the projection of $\vec{b}$ onto W.

$$PROJ_W\vec{b} = PROJ_{\vec{v}_1}\vec{b} + PROJ_{\vec{v}_2}\vec{b} = [-0.160,-0.524,-0.777,-0.937],$$

where $\vec{v}_1$ and $\vec{v}_2$ are the two orthogonal basis vectors of W. The distance from $\vec{b}$ to W is then the length of the vector $\vec{b} - PROJ_W\vec{b}$. The length of this vector is 5.022

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3. (25) Let $V = \{p \in P_3 \,:\, p(0) = 0\,\}$.

(a) Find a basis for V.

Ans: A basis for V is $\{t,\ t^2\,\}$

(b) What is the dimension of V?

Ans: Two

(c) Only one of the following sets is also a basis for V. Which one?

$$\{1, t, t^2\},\ \{t^2 - 1, t\},\ \{t^2 - 2t, t + t^2\},\ \{t^2\}$$

Ans: The first and fourth answers are eliminated since these sets do not contain two vectors. The third set is eliminated since $t^2 - 1$ is not in V. Thus, $\{t^2 - 2t, t + t^2\}$ must be the basis.

(d) If B_1 is the basis you found in part (a) and B_2 is the basis you picked in part (c), what is the change of basis matrix, P, such that $[\vec{x}]_{B_1} = P[\vec{x}]_{B_2}$?

Ans: $P = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$

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4. (30) Let $A = \begin{bmatrix} 1 & 3 \\ 1 & 6 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$. The system of equations $A\vec{x} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$ does not have a solution. By using the QR factorization of the matrix A, find the least squares solution of this equation. Be sure to explain what you are doing in terms of the mathematics, not HP key strokes.

Ans: We need to construct the matrix Q. Its columns are an orthonormal basis for the column space of A. Since $Q^TQ = I$, we also have $R = Q^TA$.

$$Q = \begin{bmatrix} 0.5 & 0.183 \\ 0.5 & 0.730 \\ 0.5 & -0.548 \\ 0.5 & -0.365 \end{bmatrix}, \qquad R = \begin{bmatrix} 2 & 4 \\ 0 & 5.477 \end{bmatrix}, \qquad Q^T \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.365 \end{bmatrix}.$$

To find the least squares solution to $A\vec{x} = \vec{b}$, we solve the equation $R\vec{x} = Q^T\vec{b}$. Thus, the least squares solution to our original equation is $\vec{x} = (0.1333, -0.0667)$.

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5. (20) Let $V = R^2$. Define the following non-standard inner product on V,

$$\vdash(\vec{x}, \vec{y})\dashv = 5x_1y_1 - x_1y_2 - x_2y_1 + 5x_2y_2 \,.$$

(a) Using the standard basis of V construct the matrix AA, where $\vdash(\vec{x}, \vec{y})\dashv = < [\vec{x}], AA[\vec{y}] >$. As usual $[\vec{x}]$ denotes the coordinates of the vector $\vec{x}$, and $< \cdot, \cdot >$ denotes the standard inner product.

Ans: The $i-j$ entry of the matrix AA equals $\vdash(\vec{x_i}, \vec{x_j})\dashv$, where x_i donotes the i^{th} basis vector. The easiest basis to work with is $\{(1,0),\ (0,1)\}$. $\vdash(\vec{e_1}, \vec{e_1})\dashv = 5$, $\vdash(\vec{e_1}, \vec{e_2})\dashv = -1$ $\vdash(\vec{e_2}, \vec{e_1})\dashv = -1$, $\vdash(\vec{e_2}, \vec{e_2})\dashv = 5$. Thus $AA = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$.

(b) Find the eigenvalues and eigenvectors of the matrix AA.

Ans: The eigenvalues and eigenvectors of AA are:

$$4,\ (1,1) \text{ and } 6,\ (1,-1)\,.$$

(c) What geometrical property do the eigenvectors of AA possess. Use the standard inner product for this.

Ans: They are perpendicular.

(d) Using what you know about the eigenvalues and eigenvectors of AA show that $\vdash(\vec{x}, \vec{x})\dashv \geq 0$ and if $\vdash(\vec{x}, \vec{x})\dashv = 0$, then $\vec{x} = \vec{0}$.

Ans: Let $\vec{v}_1$ and $\vec{v}_2$ be the two eigenvectors of AA. Let $\vec{x}$ be any vector in R^2. Then we have $[\vec{x}] = c_1\vec{v}_1 + c_2\vec{v}_2$. Note: it is the coordinates of $\vec{x}$ with respect to the standard basis that we write as a linear combination of the eigenvectors of the matrix AA. Then we have

$$\begin{aligned} \vdash(\vec{x}, \vec{x})\dashv &= < [\vec{x}], AA[\vec{x}] > = < c_1\vec{v}_1 + c_2\vec{v}_2, c_1AA\vec{v}_1 + c_2AA\vec{v}_2 > \\ &= 4c_1^2 < \vec{v}_1, \vec{v}_1 > + 6c_1c_2 < \vec{v}_1, \vec{v}_2 > + 4c_1c_2 < \vec{v}_2, \vec{v}_1 > + 6c_2^2 < \vec{v}_2, \vec{v}_2 > \\ &= 4c_1^2 < \vec{v}_1, \vec{v}_1 > + 6c_2^2 < \vec{v}_2, \vec{v}_2 > \,. \end{aligned}$$

Thus, the nonstandard inner product of a vector with itself is a sum of squares, hence nonnegative. Moreover, the only way this sum can equal zero is if both constants c_1 and c_2 equal zero. That is, the vector $\vec{x}$ equals the zero vector.
