

1. (25) An experiment leads to the following data.

x_i	1	1.5	2	2.5
y_i	-2	0	1	-1

- (a) Show that there is no first degree polynomial which fits this data? That is, no polynomial of the form $p(x) = a_0 + a_1x$ satisfies $p(x_i) = y_i$ for $1 \leq i \leq 4$.

Ans: The coefficient matrix,

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1.5 \\ 1 & 2 \\ 1 & 2.5 \end{bmatrix}$$

has rank equal to 2 and the augmented matrix has rank equal to 3. Thus, the system of equations associated with this problem does not have a solution.

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- (b) Find that first degree polynomial which is the least squares best fit solution, by use of the QR factorization technique.

Ans: We first find the QR factorization of the matrix A . It is

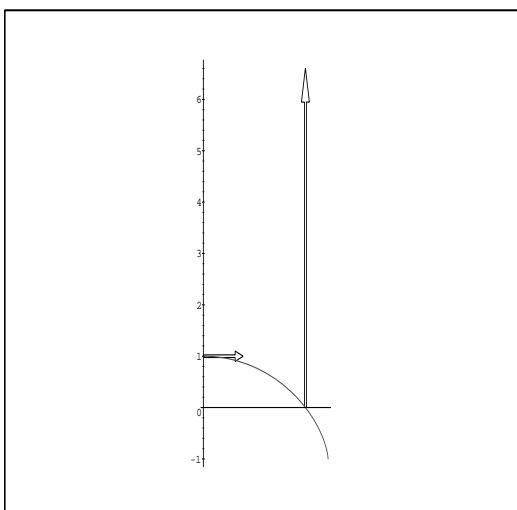
$$Q \approx \begin{bmatrix} 0.5 & -0.671 \\ 0.5 & -0.224 \\ 0.5 & 0.224 \\ 0.5 & 0.671 \end{bmatrix}, \quad R \approx \begin{bmatrix} 2 & 3.5 \\ 0 & 1.118 \end{bmatrix}.$$

Remember that the columns of Q form an orthonormal basis for the column space of A , and $R = Q^T A$. We next solve the equation $R\vec{x} = Q^T \vec{b}$, where $\vec{b} = [-2, 0, 1, -1]^T$. The solution to this equation is $a_0 = -1.9$ and $a_1 = 0.8$. Thus, the desired first degree polynomial is $p(x) = -1.9 + 0.8x$.

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2. (25) Let $\gamma(t) = (t + \sin t, \cos t)$, for $0 \leq t \leq \pi$, and $F(x, y) = (y^2, x^2)$ represent a path in R^2 and a force field respectively.

- (a) Sketch the curve γ , and on your sketch draw the force vector at the points $\gamma(0)$ and $\gamma(\pi/2)$.

Ans:



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- (b) Set up the integral whose value equals the work done by the force F on a particle which moves along the curve. Be sure that all substitutions have been done. You do not have to evaluate the integral.

Ans: Work = $\int_0^\pi (\cos^2 t, (t + \sin t)^2) \cdot (1 + \cos t, -\sin t) dt$.

- (c) Set up the integral whose value equals the flux of the force in the downward direction along this curve. Be sure that all substitutions have been done. You do not have to evaluate the integral.

Ans: Flux = $\int_0^\pi (\cos^2 t, (t + \sin t)^2) \cdot (\sin t, 1 + \cos t) dt$.

3. (25) Let $x = u^2 - v^2$, $y = 2uv$, $1 \leq u \leq 2$, $1 \leq v \leq 2$. This rectangle in u - v space will be referred to as R . Denote this transformation by T . That is, $T(u, v) = (u^2 - v^2, 2uv)$.

- (a) Show that this is an orthogonal transformation. That is, first find the vectors \vec{e}_u and \vec{e}_v , which are the unit vectors pointing in the direction of increasing u and v respectively, and then show that these two vectors are perpendicular to each other at every point in $T(R)$.

Ans: $\frac{\partial T}{\partial u} = (2u, 2v)$, $\frac{\partial T}{\partial v} = (-2v, 2u)$. Thus, $\vec{e}_u = (u, v)/\sqrt{u^2 + v^2}$ and $\vec{e}_v = (-v, u)/\sqrt{u^2 + v^2}$. The dot product of these two vectors is zero. Thus, they are perpendicular.

- (b) Find the coordinates of the gradient of any function, f , with respect to the vectors \vec{e}_u \vec{e}_v . Be sure that all derivatives are in terms of the u and v variables.

Ans: Let $\mathcal{V} = \{(u, v), (-v, u)\} = \{\vec{x}_u, \vec{x}_v\}$. Thus, $\vec{x}_u = \sqrt{u^2 + v^2}\vec{e}_u$ and $\vec{x}_v = \sqrt{u^2 + v^2}\vec{e}_v$. If P denotes the change of basis matrix which converts coordinates with respect to the basis \mathcal{V} into coordinates with respect to the standard basis, then $P = \begin{bmatrix} u & -v \\ v & u \end{bmatrix}$. An easy calculation using the chain rule shows that

$$\begin{bmatrix} \partial f / \partial u \\ \partial f / \partial v \end{bmatrix} = 2P^T \begin{bmatrix} \partial f / \partial x \\ \partial f / \partial y \end{bmatrix}$$

Putting these results together, we have:

$$\begin{aligned} [\nabla f]_{\mathcal{V}} &= P^{-1}[\nabla f]_{St} = P^{-1}(1/2)(P^T)^{-1} \begin{bmatrix} \partial f / \partial u \\ \partial f / \partial v \end{bmatrix} \\ &= (1/2) \begin{bmatrix} u^2 + v^2 & 0 \\ 0 & u^2 + v^2 \end{bmatrix}^{-1} \begin{bmatrix} \partial f / \partial u \\ \partial f / \partial v \end{bmatrix} \\ \text{Thus, } \nabla f &= \frac{1}{2\sqrt{u^2 + v^2}} \frac{\partial f}{\partial u} \vec{e}_u + \frac{1}{2\sqrt{u^2 + v^2}} \frac{\partial f}{\partial v} \vec{e}_v. \end{aligned}$$

4. (25) let $f(x, y) = 4x^2 + 5y^2 - x + 3y$. Find the maximum and minimum values of f on the set $\Omega = \left\{ (x, y) : \frac{x^2}{4} + \frac{y^2}{16} \leq 1 \right\}$.

Ans: The gradient of f is $\nabla f = (8x - 1, 10y + 3)$. It is zero at the point $(1/8, -3/10)$, which is in the region Ω . The value of the function at this point is $f(1/8, -3/10) = -0.5125$. It turns out that this is the minimum value of f on the region Ω . To determine the locations of possible extrema on the boundary of Ω the method of Lagrange multipliers is used. The constraint function is $g(x, y) = x^2/4 + y^2/16$, and the constraint is $g(x, y) = 1$. The vector equation $\nabla f + \lambda \nabla g = \vec{0}$, leads to the following equations; $8x - 1 + \lambda x/2 = 0$, $10y + 3 + \lambda y/8$. Solve each equation for λ in terms of x and y , equate these two expressions, and then solve for y in terms of x . That is,

$$\begin{aligned} \lambda &= \frac{2 - 16x}{x} = \frac{-24 - 80y}{y} \implies \\ y &= \frac{-12x}{1 + 32x} \end{aligned}$$

Substituting this into the constraint equation we have:

$$\frac{x^2}{4} + \frac{1}{16} \left(\frac{-12x}{1 + 32x} \right)^2 = 1.$$

Solving this equation for x , there are four roots, we construct the following table:

x	-1.9909	-0.0345	-0.02857	1.99145
y	-0.38097	-3.99951	3.9996	-0.3692
f	17.4285	68.017	92.014	13.446

Thus, the largest and smallest values of f on the set Ω are approximately 92.014 and -0.5125 respectively.
