

1. (30) Let  $A$  be a  $3 \times 3$  matrix. The matrix  $U$  in the  $LU$  factorization of  $A$  equals

$$U = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{bmatrix}.$$

Moreover we know that the multipliers of  $A$  equal

$$l_{21} = -2, l_{31} = 1, \text{ and } l_{32} = 5.$$

- a. What must  $A$  equal?

The multipliers tell us that  $L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 5 & 1 \end{bmatrix}$ . Thus,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 0 \\ -4 & -1 & 5 \\ 2 & 6 & 28 \end{bmatrix}$$

- b. Find  $L^{-1}$ .

We could compute  $L^{-1}$  directly from  $L$ , but it's probably easier to use the elimination matrices.

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -11 & -5 & 1 \end{bmatrix}$$

2. (30) Let  $A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & 3 & 0 & 1 \\ 1 & 4 & 6 & 8 \end{bmatrix}$ .

a. Find the reduced row echelon form of  $A$ .

$$R = \begin{bmatrix} 1 & 0 & 0 & -\frac{17}{20} \\ 0 & 1 & 0 & \frac{9}{10} \\ 0 & 0 & 1 & \frac{7}{8} \end{bmatrix}$$

b. Find a basis for the null space of  $A$ .

$A$  has rank 3, thus its null space has dimension 1. A basis for the null space is:

$$\text{basis} = \left\{ \left[ \frac{17}{20}, \frac{-9}{10}, \frac{-7}{8}, 1 \right]^T \right\}$$

c. Find a basis for the column space of  $A$ .

The column space of  $A$  has dimension 3. Since the first 3 columns of  $R$  form a basis for its column space the first 3 columns of  $A$  must form a basis for its column space. Thus,

$$\text{basis of } C[A] = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix} \right\}.$$

Note, since the column space is all of  $R^3$ , any basis of  $R^3$  is also a basis of  $C[A]$ .

d. Find a basis for the row space of  $A$ .

Since the row space of  $R$  equals the row space of  $A$ , we can use the three non-zero rows of  $R$  as our basis.

$$\text{basis of } R[A] = \left\{ \begin{bmatrix} \frac{17}{20} \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{-9}{10} \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \frac{-7}{8} \\ 1 \end{bmatrix} \right\}.$$

e. Find a basis for the null space of  $A^T$ .

Since the rank of  $A^T$  is 3, and  $A^T$  has three columns, the dimension of  $A^T$ 's null space is zero. Thus,  $N[A^T] = \{\vec{0}\}$ , and there is no basis.

f. Does the equation  $A\vec{x} = \begin{bmatrix} 1 & 1 & 1 & -3 \end{bmatrix}^T$  have a solution?

Since  $A$  is a  $3 \times 4$  matrix  $A\vec{x}$  will be in  $R^3$  not  $R^4$ . So there is no solution.

g. Does the equation  $A\vec{x} = \begin{bmatrix} 3 & 2 & 100 \end{bmatrix}^T$  have a solution?

$A\vec{x} = \vec{b}$  has a solution if and only if  $\vec{b} \in C[A]$ , and for this  $A$  its column space is all of  $R^3$ , so yes,  $A\vec{x} = \begin{bmatrix} 3 & 2 & 100 \end{bmatrix}^T$  has a solution.

3. (30) Let  $V$  be a vector space, and  $S$  a subset of  $V$ .

a. Define,  $S_p[S]$ , the span of  $S$ .

The span of the set  $S$  is the collection of all possible linear combinations of the vectors in the set  $S$ . That is, if  $S = \{\vec{x}_1, \dots, \vec{x}_k\}$ , then

$$S_p[S] = \left\{ \sum_{i=1}^k \alpha_i \vec{x}_i : \text{for any } \alpha_i \in R \right\}$$

b. Let  $S = \{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$  and  $T = \{\vec{x}_1, \alpha\vec{x}_1 + \vec{x}_2, \dots, \vec{x}_k\}$ , where  $\alpha$  is an arbitrary real number. Show that  $S_p(S) = S_p(T)$ .

Suppose that  $\vec{x} \in S_p[T]$ , then there are constants  $c_i$  such that

$$\begin{aligned} \vec{x} &= c_1\vec{x}_1 + c_2(\alpha\vec{x}_1 + \vec{x}_2) + c_3\vec{x}_3 + \dots + c_k\vec{x}_k \\ &= (c_1 + \alpha c_2)\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 + \dots + c_k\vec{x}_k. \end{aligned}$$

Thus,  $\vec{x} \in S_p[S]$ . Conversely if  $\vec{x} \in S_p[S]$ , there are constants  $c_i$  such that

$$\begin{aligned} \vec{x} &= c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 + \dots + c_k\vec{x}_k \\ &= c_1\vec{x}_1 - \alpha c_2\vec{x}_1 + \alpha c_2\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 + \dots + c_k\vec{x}_k \\ &= (c_1 - \alpha c_2)\vec{x}_1 + c_2(\alpha\vec{x}_1 + \vec{x}_2) + c_3\vec{x}_3 + \dots + c_k\vec{x}_k. \end{aligned}$$

Thus,  $\vec{x} \in S_p[T]$ , and we may conclude that  $S$  and  $T$  have the same spans.

c. What does it mean to say  $S = \{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$  is linearly independent.

A set of vectors is linearly independent if the only linear combination of those vectors, which equals the zero vector is the trivial one. That is, if there are constants  $c_i$  such that

$$c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 + \dots + c_k\vec{x}_k = \vec{0},$$

then  $c_i = 0$  for each  $i$ .

d. Let  $S = \{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$  be linearly independent. Suppose  $\vec{y} \in S_p[S]$ . Show that the set  $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k, \vec{y}\}$  is linearly dependent?

Since  $\vec{y} \in S_p[S]$ , there are constants  $c_i$  such that

$$\begin{aligned} c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 + \dots + c_k\vec{x}_k &= \vec{y} \\ c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 + \dots + c_k\vec{x}_k - \vec{y} &= \vec{0}. \end{aligned}$$

Since not all of the coefficients in the last equation are zero (the coefficient of  $\vec{y}$  is  $-1$ ), the set of vectors  $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k, \vec{y}\}$  is linearly dependent.

4. (10)

a. Let  $W = \left\{ \vec{x} = [x_1, x_2, x_3, x_4]^T \in \mathbb{R}^4 : x_1 - x_2 + x_3 = 1 \right\}$ . Is  $W$  a subspace of  $\mathbb{R}^4$ ?

$W$  is not a subspace. The easiest way to see this is to observe that  $\vec{0} \notin W$ .

b. Let  $W = \left\{ f \in F : \int_{-1}^1 f(x) dx = 0 \right\}$ . Is  $W$  a subspace of  $F$ , where  $F$  is the vector space of real valued continuous functions  $f$  with domain all real numbers.

$W$  is a subspace. It is not empty as the zero function is in  $W$ . Moreover if  $f$  and  $g$  belong to  $W$ , and  $\alpha$  is any real constant we have

$$\int_{-1}^1 (f(x) + g(x)) dx = \int_{-1}^1 f(x) dx + \int_{-1}^1 g(x) dx = 0 + 0 = 0$$

$$\int_{-1}^1 \alpha f(x) dx = \alpha \int_{-1}^1 f(x) dx = \alpha 0 = 0 .$$

Since  $W$  is nonempty and closed under vector addition and scalar multiplication,  $W$  is a subspace.