Math 323

Solutions Exam 2

Nov. 6, 2009

1. (25) Suppose you have the following data points $\{(0,2),(1,3),(2,1),(3,2)\}$. Find an equation of the straight line that best fits this data.

We are looking for a line of the form $y = dx + c$ such that $y_i = dx_i + c$ for each of the data points (x_i, y_i). The coefficient matrix of this system of equations, A, and the forcing term, $\vec{b}$ are

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}, \vec{b} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix}.$$

The normal equations $A^T A \vec{x} = A^T \vec{b}$, with $\vec{x} = \begin{bmatrix} c & d \end{bmatrix}^T$ are

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} \frac{23}{10} \\ -\frac{1}{5} \end{bmatrix}$$

Thus, the line of best fit is $y = -\frac{x}{5} + \frac{23}{10}$.

2. (20) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 5 \\ 6 & 7 & 1 \end{bmatrix}$.

a. Calculate the determinant of A.

$$\det\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 5 \\ 6 & 7 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -7 \\ 0 & -5 & -17 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} -3 & -7 \\ -5 & -17 \end{bmatrix}\right)$$

$$= (-3)(-17) - (35)$$

$$= 16$$

b. What is the 2-3 entry of A^{-1}. (row 2 column 3)?

The entry in the second row and third column of A^{-1} equals

$$\frac{M_{3,2}}{\det A} = \frac{-\det\left(\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}\right)}{16} = \frac{7}{16}$$

3. (35) Let $W = \{\vec{x} = (x_1, x_2, x_3, x_4) : x_1 - x_2 + 2x_3 - x_4 = 0\}$.

a. Find an orthonormal basis of W.

W is the null space of the 1×4 matrix $A = \begin{bmatrix} 1 & -1 & 2 & -1 \end{bmatrix}$. The rank of A is 1, so its null space has dimension 3. A basis for W is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \right\}.$$

Using the Gram-Schmidt method to construct an orthonormal basis, we have

$$\vec{q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{-2}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{q}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/6 \\ -1/6 \\ 1/3 \\ 1 \end{bmatrix}$$

$$\vec{q}_3 = \frac{1}{\sqrt{42}} \begin{bmatrix} 1 \\ -1 \\ 2 \\ 6 \end{bmatrix}$$

b. Find an orthonormal basis of $W^{\perp}$.

Since $W^{\perp}$ is the orthogonal complement of W, which is the null space of A, $W^{\perp}$ is the row space of A. A basis for the row space of A is $\begin{bmatrix} 1 & -1 & 2 & -1 \end{bmatrix}^T$. Thus, an orthornormal basis is the single vector $\frac{1}{\sqrt{7}} \begin{bmatrix} 1 & -1 & 2 & -1 \end{bmatrix}^T$.

4. (20) Let A be an $m \times n$ matrix.

a. Show that if the columns of A are linearly independent then the null space of A contains only the zero vector.

Denote the n columns of A by $\vec{c}_1, \cdots, \vec{c}_n$, then if $\vec{x} = \left[x_1, \cdots, x_n\right]^T$ is in the null space of A we have

$$\vec{0} = A\vec{x} = x_1\vec{c}_1 + \cdots + x_n\vec{c}_n \ .$$

Since the columns of A are linearly independent each $x_i = 0$, or $\vec{x} = \vec{0}$. That is $N(A) = \left\{\vec{0}\right\}$.

b. Show that $N(A^TA) = N(A)$.

Clearly $N(A) \sqsubseteq N(A^TA)$. So suppose $\vec{x} \in N(A^TA)$. Then we have

$$0 = \vec{x}^TA^TA\vec{x} = (A\vec{x})^TA\vec{x} \ .$$

That is $\|A\vec{x}\| = 0$, which means that $A\vec{x} = \vec{0}$, or $\vec{x} \in N(A)$, and the two null spaces are equal.

c. Use parts a. and b. to show that if the columns of A are linearly independent, then A^TA is invertible.

Assuming that the columns of A are linearly independent part a. tells us that the null space of A consists only of the zero vector, and part b. now tells us that the null space of A^TA is just the zero vector. That is, A^TA is a square matrix with a full set of pivots. Thus, A^TA is invertible.