As usual, a bald answer is worth nothing. No explanation, no credit, and no calculators allowed.

1. (40) The separate parts of this question all refer to the following system of equations:

$$2x_1 + 3x_2 - 5x_4 = 1$$
$$x_1 + x_2 + x_3 + x_4 = -1$$
$$x_1 - x_2 + 5x_3 = 2\ .$$

a. If we write this system in the form $A\vec{x} = \vec{b}$, what are A, $\vec{x}$, and $\vec{b}$?

$$A = \begin{bmatrix} 2 & 3 & 0 & -5 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 5 & 0 \end{bmatrix}, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \text{ and } \vec{b} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

b. Find a basis for the null space of A.

The augmented matrix of this system is

$$\begin{bmatrix} 2 & 3 & 0 & -5 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ 1 & -1 & 5 & 0 & 2 \end{bmatrix},$$

and its reduced row echelon form is

$$\begin{bmatrix} 1 & 0 & 3 & 0 & \frac{4}{5} \\ 0 & 1 & -2 & 0 & -\frac{6}{5} \\ 0 & 0 & 0 & 1 & -\frac{3}{5} \end{bmatrix}.$$

Thus, a basis for the null space of A is $\left\{\left[-3, 2, 1, 0\right]^T\right\}$.

c. Find all solutions of this system.

A particular solution to this system is $\vec{b} = \left[\frac{4}{5}, -\frac{6}{5}, 0, -\frac{3}{5}\right]$. Thus, any solution can be written as

$$\alpha\begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{5} \\ -\frac{6}{5} \\ 0 \\ -\frac{3}{5} \end{bmatrix} \text{ for some constant } \alpha.$$

2. (45) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}$.

a. Let $Q[\vec{x}]$ denote the quadratic form determined by A.What is $Q[(-1,2,7)]$?

$$Q[(-1,2,7)] = \begin{bmatrix} -1 & 2 & 7 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 5 & 13 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 7 \end{bmatrix}$$

$$= 77 .$$

b. Find the LDL^T factorization of A.

We see that $l_{2,1} = 2$ and $l_{3,1} = 3$. Performing these two eliminations we have A rwo equivalent to

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -5 \\ 0 & -5 & -7 \end{bmatrix}.$$

Thus, $l_{3,2} = 5/4$, and $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 5/4 & 1 \end{bmatrix}$, and $A = LU$, where $U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -5 \\ 0 & 0 & -\frac{3}{4} \end{bmatrix}$.

Thus,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 5/4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -5 \\ 0 & 0 & -\frac{3}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 5/4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -3/4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5/4 \\ 0 & 0 & 1 \end{bmatrix}.$$

c. Use the LDL^T factorization found in part b. to express $Q(\vec{x})$ as a sum of squares.

$$
\begin{aligned}
Q(\vec{x}) &= \vec{x}^T A \vec{x} = \vec{x}^T L D L^T \vec{x} \\
&= (L^T\vec{x})^T D (L^T\vec{x}) \\
&= \left[x_1 + 2x_2 + 3x_3,\ x_2 + \frac{5}{4}x_3,\ x_3 \right] \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -3/4 \end{bmatrix} \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ x_2 + \frac{5}{4}x_3 \\ x_3 \end{bmatrix} \\
&= (x_1 + 2x_2 + 3x_3)^2 - 4\left(x_2 + \frac{5}{4}x_3\right)^2 - \frac{3}{4}x_3^2 \,.
\end{aligned}
$$

d. Is the matrix A positive definite, negative definite or neither?

A is neither postive definite nor negative definite as there are both positive and negative pivots.

e. Explain why a symmetric matrix is positive definite if all of its pivots are positive.

When writing the quadratic form determined by a symmetric matrix. If the LDL^T factorization of A is used, then we see that the constants multiplying the squares are the pivots of A. So if all of the pivots are positive then the quadratic form is definite and if $Q(\vec{x}) = 0$, then we must have $L^T\vec{x} = \vec{0}$, and since L^T is invertible $\vec{x} = \vec{0}$, and the quadratic form is positive definite, which is the definition of A being positive definite.

3. (30) Let $A = \frac{1}{6}\begin{bmatrix} 2 & -4 & 4 \\ 1 & 4 & -1 \\ 1 & -2 & 5 \end{bmatrix}$.

a. A has eigenvalues 1/3, 1/2, and 1. Find a basis of R^3, which consists of eigenvectors of A.

Let $\hat{A} = 6A$. Then for $\lambda = 1/3$:

$$\begin{aligned} A - \frac{1}{3}I &= \frac{1}{6}\left(\hat{A} - 2I\right) \\ &= \frac{1}{6}\begin{bmatrix} 0 & -4 & 4 \\ 1 & 2 & -1 \\ 1 & -2 & 3 \end{bmatrix} \\ &\approx \frac{1}{6}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

An eigenvector associated to $\lambda = 1/3$ is $\left(-1, 1, 1\right)$.

For $\lambda = 1/2$ we have

$$\begin{aligned} A - \frac{1}{2}I &= \frac{1}{6}\left(\hat{A} - 3I\right) \\ &= \frac{1}{6}\begin{bmatrix} -1 & -4 & 4 \\ 1 & 1 & -1 \\ 1 & -2 & 2 \end{bmatrix} \\ &\approx \frac{1}{6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

An eigenvector associated to $\lambda = 1/2$ is $\left(0, 1, 1\right)$.

For $\lambda = 1$ we have

$$A - I = \frac{1}{6}\left(\hat{A} - 6I\right)$$
$$= \frac{1}{6}\begin{bmatrix} -4 & -4 & 4 \\ 1 & 2 & -1 \\ 1 & -2 & -1 \end{bmatrix}$$
$$\approx \frac{1}{6}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

An eigenvector associated to $\lambda = 1$ is $\left(1, 0, 1\right)$.

b. Let $\vec{x} = \left[1, 1, 2\right]^T$. Determine $\lim\limits_{n \to \infty} A^n \vec{x}$.

Write $\vec{x}$ as a linear combination of the three eigenvectors.

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}. \text{ Thus,}$$

$$A^n\vec{x} = A^n\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + A^n\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \left(\frac{1}{2}\right)^n\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Taking the limit as $n \to \infty$ we have

$$\lim_{n \to \infty} A^n\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

c. Find a matrix S such that $D = S^{-1}AS$ is a diagonal matrix. What is D ?

Set $S = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Then $D = S^{-1}AS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$.

4. (15) Let T be a linear transformation from the vector space V into the vector space W. Let $\left\{\vec{v}_1, \cdots, \vec{v}_n\right\} = B_V$ and $\left\{\vec{w}_1, \cdots, \vec{w}_m\right\} = B_W$ be bases of V and W respectively.

a. Prove, using the defintion of a linear transformation, that $T\left(\vec{0}\right) = \vec{0}$.

$$T\left(\vec{0}\right) = T\left(\vec{0} + \vec{0}\right) = T\left(\vec{0}\right) + T\left(\vec{0}\right).$$

The last equation implies, since $T\left(\vec{0}\right)$ has an additive inverse in W, that $T\left(\vec{0}\right) = \vec{0}$.

b. Define the matrix representation, A, of T with respect to the bases B_V and B_W. That is, you need to say two things: one what the columns of A consist of, and two, what equation A must satisfy with respect to the coordinates of $\vec{x} \in V$ and $T(\vec{x}) \in W$.

The equation, which A must satisfy, is that the coordinates of $T(\vec{x})$ with respect to the basis B_W equal A times the coordinates of $\vec{x}$ with respect to the basis B_V.

$$[T(\vec{x})]_{B_W} = A[\vec{x}]_{B_V} .$$

The i^{th} column of A consists of the coordinates of $T(v_i)$ with respect to the basis B_W.

5. (20) Let $W = \left\{\vec{x} = \left[x_1, x_2, x_3, x_4\right] \in R^4 : x_1 + x_2 + x_3 + x_4 = 0, x_1 - x_2 - x_3 + x_4 = 0\right\}$. Let T denote the linear transformaion of R^4 that reflects R^4 through the subspace W. Find the matrix representation of T with respect to the standard basis of R^4.

We want a nice basis of R^4. W has dimension 2 and the two vectors $v_1 = \left(1, 0, 0, -1\right)$, $\vec{v}_2 = \left(0, 1, -1, 0\right)$ form a basis for W. Since we want to reflect R^4 through W we want to extend this basis of W to a basis of R^4 that consists of vectors perpendicular to W. Two such vectors are $\vec{v}_3 = \left(1, 0, 0, 1\right)$ and $\vec{v}_4 = \left(0, 1, 1, 0\right)$. The linear transformation T then maps these vectors as follows:

$$T(\vec{v}_1) = \vec{v}_1, T(\vec{v}_2) = \vec{v}_2, T(\vec{v}_3) = -\vec{v}_3, T(\vec{v}_4) = -\vec{v}_4$$

Letting A_V denote the matrix representation of T with respect to this basis of R^4 we have

$$A_V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

The matrix $P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$ is the change of basis matrix for which $[\vec{x}]_S = P[\vec{x}]_V$. The matrices A_V and P satisfy the following equation:

$$\begin{aligned} [T(\vec{x})]_S &= P[T(\vec{x})]_V = PA_V[\vec{x}]_V \\ &= PA_VP^{-1}[\vec{x}]_S . \end{aligned}$$

Thus, the matrix representation of T with respect to the standard basis must equal

$$
\begin{aligned}
A &= PA_VP^{-1} \\
&= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$