1. (20) Let $A = \begin{bmatrix} 2 & 0 & 5 \\ 0 & 4 & 11 \\ -3 & 1 & 2 \end{bmatrix}$.

   (a) Calculate the determinant of A.

$$\det \begin{bmatrix} 2 & 0 & 5 \\ 0 & 4 & 11 \\ -3 & 1 & 2 \end{bmatrix} = \det \begin{bmatrix} 2 & 0 & 5 \\ 12 & 0 & 3 \\ -3 & 1 & 2 \end{bmatrix} = -\det \begin{bmatrix} 2 & 5 \\ 12 & 3 \end{bmatrix}$$
$$= -(-54) = 54$$

   (b) What is the $2,3$ entry of A^{-1}?

$$(-1)^{2+3} \frac{\det M_{3,2}}{\det A} = -\frac{\det \begin{bmatrix} 2 & 5 \\ 0 & 11 \end{bmatrix}}{54} = -\frac{22}{54} = -\frac{11}{27}$$

2. (10) Let $L : V \to W$ be a linear transformation. Suppose A is the matrix representation of L with respect to bases B_V and B_W of V and W respectively.

   (a) What is the relationship between the kernel of L and the null space of A?

   The basic equation describing the relationship between L and A is $[L(\boldsymbol{x})]_{B_W} = A[\boldsymbol{x}]_{B_V}$. This tells us that $\boldsymbol{x}$ is in the kernel of L ($L(\boldsymbol{x}) = \mathbf{0}$) if and only if the coordinates of $\boldsymbol{x}$ with respect to the basis B_V are in the null space of A.

   (b) What is the relationship between the range of L and the column space of A?

   A vector $\hat{\boldsymbol{y}}$ is in the column space of A if and only if there is a vector $\hat{\boldsymbol{x}}$ such that $A\hat{\boldsymbol{x}} = \hat{\boldsymbol{y}}$. So we see that a vector $\boldsymbol{y}$ is in the range of L if and only if the coordinates of $\boldsymbol{y}$ with respect to the basis B_W are in the column space of A.

3. (25) Let $L : \mathbb{P}_2 \to \mathbb{P}_2$ be a linear transformation, where $L(\boldsymbol{p}) = \boldsymbol{p}' - 2\boldsymbol{p}$. Let $\mathcal{S} = \{1, t, t^2\}$.

(a) Find the matrix representation of L with respect to the basis $\mathcal{S}$.

$$L(\mathbf{1}) = -\mathbf{2} \qquad L(\boldsymbol{t}) = \mathbf{1} - 2\boldsymbol{t} \qquad L(\boldsymbol{t}^2) = 2\boldsymbol{t} - 2\boldsymbol{t}^2 \,.$$

Thus, the matrix representation equals $\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

(b) Is L one-to-one?

Since the rank of the matrix is 3, this tells us the dimension of its nullspace is 0, which means the kernel of L has dimension 0. Thus, L is one-to-one.

(c) Is L onto?

Since the dimension of $\mathbb{P}_2$ is 3, we know the column space of the matrix representation is $\mathbb{R}^3$. Thus, L must be onto $\mathbb{P}_2$.

4. (20) Let $\mathcal{K} = \{1 - t + t^2, -2t + 3t^2, 1 + t^2\}$.

(a) Find the change of basis matrix P such that $[\boldsymbol{p}]_{\mathcal{S}} = P[\boldsymbol{p}]_{\mathcal{K}}$, where $\mathcal{S} = \{1, t, t^2\}$.

$$P = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

(b) If A is the matrix representation of a linear transformation L from $\mathbb{P}^2 \to \mathbb{P}^2$ with respect to the basis $\mathcal{S}$, what is the matrix representation of L with respect to the basis $\mathcal{K}$. Your answer should be in terms of A and P.

Let $\hat{A}$ denote the matrix we're looking for. Then we have

$$[L(\boldsymbol{x})]_{\mathcal{K}} = P^{-1}[L(\boldsymbol{x})]_{\mathcal{S}} = P^{-1}A[\boldsymbol{x}]_{\mathcal{S}} = P^{-1}AP[\boldsymbol{x}]_{\mathcal{K}} \,.$$

Thus, $\hat{A} = P^{-1}AP$.

5. (25) Let $P = \begin{bmatrix} 2 & 3 & 5 \\ 0 & -6 & 7 \\ 0 & 4 & 11 \end{bmatrix}$, and suppose $\mathcal{F} = \{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3\}$ is a basis of $\mathbb{R}^3$.

(a) Suppose P is the matrix representation of a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$, with respect to the basis $\mathcal{F}$. Find $L(\boldsymbol{f}_2)$ in terms of the basis vectors $\boldsymbol{f}_i$.

The coordinates of $\boldsymbol{f}_2$ with respect to the basis $\mathcal{F}$ are $(0, 1, 0)^T$. Thus, the coordinates of $L(\boldsymbol{f}_2)$ are given by

$$[L(\boldsymbol{f}_2)]_{\mathcal{F}} = \begin{bmatrix} 2 & 3 & 5 \\ 0 & -6 & 7 \\ 0 & 4 & 11 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 4 \end{bmatrix},$$

and

$$L(\boldsymbol{f}_2) = 3\boldsymbol{f}_1 - 6\boldsymbol{f}_2 + 4\boldsymbol{f}_3 .$$

(b) Suppose P is the change of basis matrix such that $[\boldsymbol{x}]_{\mathcal{S}} = P[\boldsymbol{x}]_{\mathcal{F}}$. Then $\boldsymbol{f}_3 =?$

The coordinates of $\boldsymbol{f}_3$ with respect to the basis $\mathcal{F}$ are $(0, 0, 1)^T$. Thus, the coordinates of $\boldsymbol{f}_3$ with respect to the standard basis are

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & -6 & 7 \\ 0 & 4 & 11 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix}.$$

Thus, $\boldsymbol{f}_3 = (5, 7, 11)$.

(c) What is the first column of P^n.

Notice that the first column of P has a 2 in the 1,1 position and zeros elsewhere. Thus, if A is any matrix, the first column of AP must be 2 times the first column of A. Hence the first column of P^n must equal

$$\begin{bmatrix} 2^n \\ 0 \\ 0 \end{bmatrix}.$$