1. (40) $\mathcal{W} = \{\boldsymbol{x} \in \mathbb{R}^4 : x_1 + x_3 = 0,\ x_1 + x_4 = 0\}$. For the following questions use the standard inner product on $\mathbb{R}^4$.

   (a) Show that $\mathcal{W}$ is a subspace of $\mathbb{R}^4$.

   Let $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$. The set W is the null space of the matrix A, and hence it must be a subspace.

   (b) Find an orthonormal basis of $\mathcal{W}$.

   Since the rank of A is two, W is a $4 - 2 = 2$ dimensional subspace of $\mathbb{R}^4$. A basis of W is given by the set $\{(0,1,0,0), (-1,0,1,1)\}$. Note that these two vectors are orthogonal. Thus, one orthonormal basis of W consists of the two vectors

$$\boldsymbol{u}_1 = (0,1,0,0), \quad \boldsymbol{u}_2 = \frac{(-1,0,1,1)}{\sqrt{3}}.$$

   (c) Find the projection of $\boldsymbol{b} = (1,1,1,1)$ onto $\mathcal{W}$.

$$\begin{aligned} \text{Proj}_{\mathcal{W}}\boldsymbol{b} &= \text{Proj}_{\boldsymbol{u}_1}\boldsymbol{b} + \text{Proj}_{\boldsymbol{u}_2}\boldsymbol{b} = 1\boldsymbol{u}_1 + \frac{1}{\sqrt{3}}\boldsymbol{u}_2 = (0,1,0,0) + \frac{1}{\sqrt{3}}\frac{(-1,0,1,1)}{\sqrt{3}} \\ &= \frac{1}{3}(-1,3,1,1) \end{aligned}$$

2. (40) Let L be a linear transformation from $\mathcal{P}_2$ into $\mathcal{P}_2$. Suppose that

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 3 & 1 \\ -1 & 0 & 5 \end{bmatrix}$$

is the matrix representation of L with respect to the standard basis of $\mathcal{P}_2$.

(a) $L(1-2t^2) = ?$

The coordinates of $1-2t^2$ with respect to the standard basis, $\{1, t, t^2\}$, are $[1, 0, -2]$. Thus, the coordinates of $L(1-2t^2) =$ with respect to the standard basis are

$$A[(1-2t^2)]_S = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 3 & 1 \\ -1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ -11 \end{bmatrix} .$$

Thus,

$$L(1-2t^2) = -1 - 4t - 11t^2 .$$

(b) What are the dimensions of the kernel of L and the range of L?

The matrix A has rank equal to 3. Thus, the range of L has dimension 3, and the kernel of L has dimension equal to 0.

(c) What is the matrix representation of L with respect to the basis $B = \{t, t^2, 1\}$ of $\mathcal{P}_2$.

Let

$$\boldsymbol{v}_1 = t, \quad \boldsymbol{v}_2 = t^2 \quad \boldsymbol{v}_3 = 1 ,$$

then $L(\boldsymbol{v}_i)$ equals

$$L(\boldsymbol{v}_1) = 3t = 3\boldsymbol{v}_1, \quad L(\boldsymbol{v}_2) = 1+t+5t^2 = \boldsymbol{v}_1+5\boldsymbol{v}_2+\boldsymbol{v}_3, \quad L(\boldsymbol{v}_3) = 1-2t-t^2 = -2\boldsymbol{v}_1-\boldsymbol{v}_2+\boldsymbol{v}_3 .$$

Thus, the matrix representation of L with respect to this second basis is

$$\begin{bmatrix} 3 & 1 & -2 \\ 0 & 5 & -1 \\ 0 & 1 & 1 \end{bmatrix} .$$

3. (30) Let $D = \{(x_i, y_i)\} = \{(-1,1),(0,1),(1,3),(4,2)\}$.

(a) Show that there is not a quadratic polynomial, $q(x) = a_0 + a_1 x + a_2 x^2$ such that $q(x_i) = y_i$ for each pair $(x_i, y_i) \in D$.

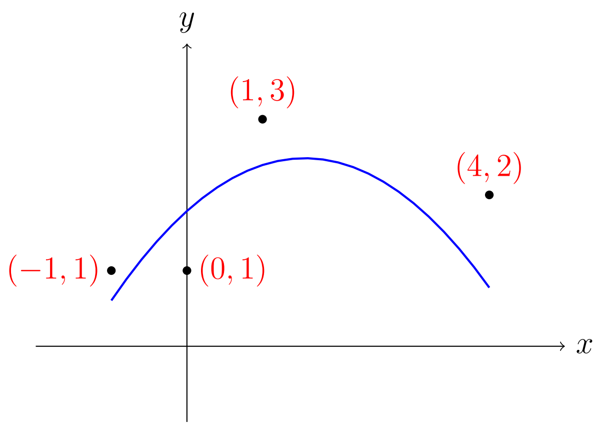
If there were such a polynomial then its coefficients a_i must satisfy the equations

$$a_0 - a_1 + a_2 = 1, \quad a_0 = 1, \quad a_0 + a_1 + a_2 = 3, \quad a_0 + 4a_1 + 16a_2 = 2\,.$$

This leads to the matrix equation

$$A\boldsymbol{b} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}.$$

A quick calculation shows that the rank of the augmented matrix is 4, but the rank of A is 3. Thus, this system has no solution. A picture demonstration that this system has no solution can be see by plotting the data points. Note the quadratic polynomial, which best fits the data is included in the plot.



It's clear that there is no parabola, which passes through these 4 points.

(b) What equations should be used to find that quadratic polynomial, $\boldsymbol{q}$, which is the best least squares fit to the data?

The system to be solved is $A\boldsymbol{x} = \text{Proj}_{\text{CS}(A)}\boldsymbol{b}$. The reason for this is that the solution to this equation is such that $A\boldsymbol{x}$ is as close as possible to $\boldsymbol{b}$. This system is equivalent to the 'normal' equations, which are $A^T A\boldsymbol{x} = A^T\boldsymbol{b}$, which leads to the following system and least squares solution

$$\begin{bmatrix} 4 & 4 & 18 \\ 4 & 18 & 64 \\ 18 & 64 & 258 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 36 \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \frac{1}{362} \begin{bmatrix} 647 \\ 324 \\ -75 \end{bmatrix} \approx \begin{bmatrix} 1.787 \\ 0.895 \\ -0.207 \end{bmatrix}$$

4. (20) Which of the following are true statements. If a statement is true supply a proof, and if it's false give a counter example.

(a) The bilinear form $\langle \boldsymbol{f}, \boldsymbol{g} \rangle = \boldsymbol{f}(0)\boldsymbol{g}(0) + \boldsymbol{f}(1)\boldsymbol{g}(1)$ is an inner product on $C[0,1]$.

This bilinear form satisfies all of the requirements to be an inner product except that it is not positive definite. Let $f(t) = \sin(\pi t)$. Then

$$\langle \boldsymbol{f}, \boldsymbol{f} \rangle = \sin^2(0) + \sin^2(\pi) = 0$$

Since the sine function is not the zero function, we do not have an inner product.

(b) Let $B = \{(1,1), (1,0), (-1,3)\}$. For any $\boldsymbol{x} \in \mathbb{R}^2$ there is only one way to write $\boldsymbol{x}$ as a linear combination of the vectors in B.

These three vectors are linearly dependent, which means that the zero vector can be written in more than one way as a linear combination of the vectors in B. So the statement is false. An example of this follows

$$\begin{aligned} \mathbf{0} &= 0(1,1) + 0(1,0) + 0(-1,3) \\ &= -3(1,1) + 4(1,0) + (-1,3) \end{aligned}$$

(c) The matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ represents a linear transformation from $\mathbb{R}^4 \to \mathbb{R}^2$, which is onto.

This too is false. A represents a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^4$.

5. (20) Let $Q(\boldsymbol{x}) = 2x_1^2 - 4x_1x_2 + 2x_2^2$.

(a) Find a symmetric matrix, A, such that $Q(\boldsymbol{x}) = \langle \boldsymbol{x}, A\boldsymbol{x} \rangle$, where $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ represents the standard inner product on $\mathbb{R}^2$.

$$\begin{aligned} Q(\boldsymbol{x}) &= 2x_1^2 - 4x_1x_2 + 2x_2^2 = x_1(2x_1 - 4x_2) + x_2(2x_2) \\ &= \langle (x_1, x_2), (2x_1 - 4x_2, 2x_2) \rangle \\ &= \langle \boldsymbol{x}, \hat{A}\boldsymbol{x} \rangle, \text{ where } \hat{A} = \begin{bmatrix} 2 & -4 \\ 0 & 2 \end{bmatrix}. \end{aligned}$$

The symmetric matrix we are looking for equals

$$A = \frac{\hat{A} + \hat{A}^T}{2} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

(b) Plot the set of points $Q(\boldsymbol{x}) = 1$. The eigenvalues and eigenvectors of A are

$$0, (1,1), \text{ and } 4, (-1,1)$$

Set $\boldsymbol{u}_1 = \frac{(1,1)}{\sqrt{2}}$ and $\boldsymbol{u}_2 = \frac{(-1,1)}{\sqrt{2}}$. These two vectors form an orthonormal basis of $\mathbb{R}^2$. Moreover $A\boldsymbol{u}_1 = \boldsymbol{0}$, and $A\boldsymbol{u}_2 = 4\boldsymbol{u}_2$. Thus, if $\boldsymbol{x} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2$, we have

$$\begin{aligned} Q(\boldsymbol{x}) &= \langle c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2, A(c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2) \rangle \\ &= \langle c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2, c_2 4\boldsymbol{u}_2 \rangle \\ &= 4c_2^2. \end{aligned}$$

Thus, the equation $Q(\boldsymbol{x}) = 1$ becomes $c_2 = \pm\frac{1}{2}$. A plot of these points is shown below.