1. (15)

a. Define the null space of an $m \times n$ matrix A.

The null space of $A = NS(A) = \left\{\vec{x} \in R^n : A\vec{x} = \vec{0}\right\}$.

b. Define what it means to say that the set of vectors $\left\{\vec{x}_1, \cdots, \vec{x}_k\right\}$ is linearly dependent.

The set $\left\{\vec{x}_1, \cdots, \vec{x}_k\right\}$ is linearly dependent if there exist constants c_i, $1 \le i \le k$, not all of which are 0 such that

$$\sum_{i=1}^{k} c_i \vec{x}_i = \vec{0}.$$

c. If $A = [a_{i,j}]$ is an $m \times n$ matrix and $B = [b_{i,j}]$ is an $n \times p$ matrix. How is the i-j entry of the matrix AB computed?

Denote the i-j entry of AB by $c_{i,j}$, then

$$c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}.$$

2. (25) Let $A = \begin{bmatrix} 2 & 4 & 2 & 8 \\ -2 & -4 & 0 & -6 \\ 4 & 8 & 4 & 16 \\ 2 & 4 & 1 & 7 \end{bmatrix}$.

a. Find the reduced row echelon form of A.

The reduced row echelon form of A equals

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

b. Find a basis for the null space of A.

Since A has rank 2 and 4 columns the null space of A has dimension 2. Moreover x_2 and x_4 are the free variables. The equations specifying x_1 and x_3 in terms of the free variables are

$$x_1 = -2x_2 - 3x_4, \quad x_3 = -x_4.$$

So a basis of $NS(A)$ is

$$\left\{(-2, 1, 0, 0), (-3, 0, -1, 1)\right\}.$$

c. Find a basis for the row space of A.

Since the row space of A is the same as the row space of its reduced row echelon form, a basis for the row space of A is

$$\left\{\begin{pmatrix} 1 & 2 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix}\right\}.$$

d. Find a basis for the column space of A.

Since the first and third columns of the reduced row echelon form of A form a basis for its row space, the corresponding columns of A form a basis for the column space of A. Thus, a basis for the column space of A is

$$\left\{ \begin{pmatrix} 2 \\ -2 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 4 \\ 1 \end{pmatrix} \right\}.$$

e. Does the equation $A\vec{x} = \left[1, 1, 1, 1\right]^T$ have a solution?

This equation has a solution if and only if the vector $\left[1, 1, 1, 1\right]^T$ is in the column space of A. That is, do there exist constants a and b such that

$$a\begin{pmatrix} 2 \\ -2 \\ 4 \\ 2 \end{pmatrix} + b\begin{pmatrix} 2 \\ 0 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

By checking the first and third rows we see that there cannot be a solution. For these rows imply

$$2a + 2b = 1$$
$$4a + 4b = 1,$$

and this is impossible.

3. (20) Determine whether or not the following statements are true or false. If true supply a proof and if false explain why the statement is false.

a. The set $\left\{ (x_1, x_2, x_3) \in R^3 : \text{ such that } x_1 + x_2 = 1 \right\}$ is a subspace of R^3.

Since the zero vector, $(0, 0, 0)$, is not in the given set, that set cannot be a subspace of R^3.

b. The set $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ spans $M_{2,2}$.

Ths set does not span $M_{2,2}$ as the vector $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ cannot be written as a linear combination of the vectors in the given set.

4. (30) Let $V = \{\vec{x} \in R^4 : x_1 + x_3 - x_4 = 0 \text{ and } x_1 + 2x_2 - 2x_3 + x_4 = 0\}$.

a. Show that V is a subspace of R^4 and that the set $S = \{(-2, 3, 2, 0), (1, -1, 0, 1)\}$ is a basis of V.

Perhaps the easiest way to verify that V is a subspace of R^4 is to realize that it is the null space of the matrix $A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 1 & 2 & -2 & 1 \end{bmatrix}$.

The two equations which describe V can be solved for x_1 and x_2 giving

$$x_1 = -x_3 + x_4$$
$$x_2 = \frac{1}{2}(3x_3 - 2x_4).$$

Thus, if $\vec{x} = (x_1, x_2, x_3, x_4) \in V$ we must have

$$\begin{aligned}\vec{x} &= \left(-x_3 + x_4, \frac{1}{2}(3x_3 - 2x_4), x_3, x_4\right) \\ &= \left(-x_3, \frac{3}{2}x_3, x_3, 0\right) + (x_4, -x_4, 0, x_4) \\ &= x_3(-1, 3/2, 1, 0) + x_4(1, -1, 0, 1) \\ &= \left(\frac{x_3}{2}\right)(-2, 3, 2, 0) + x_4(1, -1, 0, 1).\end{aligned}$$

Thus, $\{(-2, 3, 2, 0), (1, -1, 0, 1)\}$ is a spanning set of V, and since these two vectors are linearly independent the set is a basis of V.

b. If $\vec{x} = (x_1, x_2, x_3, x_4)$ is an arbitrary vector in V what are its coordinates with respect to the basis S.

From the work of part a. we see that the coordinates of $\vec{x}$ with respect to the basis S are

$$[(x_1, x_2, x_3, x_4)]_S = \left[\frac{x_3}{2}, x_4\right].$$

c. Show that the set $B = \{(1, 0, 2, 3), (0, 1, 2, 2)\}$ is a basis for V.

Since the two vectors are in V, are linearly independent, and $\dim(V) = 2$, the set B must be a basis.

d. Find the change of basis matrix P such that for vectors $\vec{x} \in V$ we have

$$[\vec{x}]_S = P[\vec{x}]_B.$$

The columns of the matrix P must be the coordinates of the vectors in B with respect to the basis S.

$$[(1, 0, 2, 3)]_S = [1, 3]$$
$$[(0, 1, 2, 2)]_S = [1, 2].$$

Thus,

$$P = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}.$$

5. (10) Let A be an $m \times n$ matrix, $\vec{x} \in R^n$, and $\vec{b} \in R^m$. Show that the equation $A\vec{x} = \vec{b}$ has a solution if and only if $\vec{b}$ belongs to the column space of A.

If the n columns of the matrix A are denoted by $\vec{C}_1, \vec{C}_2, \cdots, \vec{C}_n$, the product of $A\vec{x}$, where $\vec{x} = \left[x_1, x_2, \cdots, x_n\right]^T$, can be written

$$A\vec{x} = x_1\vec{C}_1 + x_2\vec{C}_2 + \cdots + x_n\vec{C}_n \ .$$

Thus, asking if the equation $A\vec{x} = \vec{b}$ has a solution is equivalent to asking if there are constants x_i such that

$$x_1\vec{C}_1 + x_2\vec{C}_2 + \cdots + x_n\vec{C}_n = \vec{b} \ .$$

That is, the equation has a solution if and only if $\vec{b}$ is in the column space of A.