

1. (25) Let L be a linear transformation from the vector space V to the vector space W .
- a. What is the kernel of L ?

$$\ker(L) = \{\vec{x} \in V : L(\vec{x}) = \vec{0}\}$$

- b. Show that the range of L is a subspace of W .

Since V is assumed to be nonempty, $L(V)$, which is the range of L is also nonempty. Thus, we only need to show that $L(V)$ is closed under vector addition and scalar multiplication. Suppose \vec{y}_1 and \vec{y}_2 are vectors in $L(V)$. Then there are vectors \vec{x}_1 and \vec{x}_2 , such that $L(\vec{x}_i) = \vec{y}_i$. Then we have

$$\vec{y}_1 + \vec{y}_2 = L(\vec{x}_1) + L(\vec{x}_2) = L(\vec{x}_1 + \vec{x}_2)$$

$$\alpha\vec{y}_1 = \alpha L(\vec{x}_1) = L(\alpha\vec{x}_1)$$

Thus, both $\vec{y}_1 + \vec{y}_2$ and $\alpha\vec{y}_1$ are in $L(V)$.

- c. Show that $L(\vec{0}) = \vec{0}$

For any $\vec{x} \in V$, we have

$$\vec{0} = 0L(\vec{x}) = L(0\vec{x}) = L(\vec{0})$$

- d. Show that if the kernel of $L = \{\vec{0}\}$, then L is one-to-one.

Suppose \vec{x}_1 and \vec{x}_2 are such that $L(\vec{x}_1) = L(\vec{x}_2)$. Then we must have

$$\vec{0} = L(\vec{x}_1) - L(\vec{x}_2) = L(\vec{x}_1 - \vec{x}_2).$$

Thus, since only the zero vector is in the kernel of L , we have $\vec{x}_1 - \vec{x}_2 = \vec{0}$. Hence L must be one-to-one.

2. (30) Let $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$ be two vectors in R^3 . Define the following function from $R^3 \times R^3 \rightarrow R$, by

$$\langle \vec{x}, \vec{y} \rangle = x_1y_1 + 3x_2y_2 + x_3y_3.$$

- a. Show that this function is an inner product on R^3 .

We need to verify three properties: positive definite, symmetric, and linear in either slot.

- i. $\langle \vec{x}, \vec{x} \rangle = x_1^2 + 3x_2^2 + x_3^2$. Since the x_i 's are real numbers this term is never negative, and if it does equal zero, then each $x_i = 0$, or $\vec{x} = \vec{0}$.

ii.

$$\langle \vec{x}, \vec{y} \rangle = x_1y_1 + 3x_2y_2 + x_3y_3 = y_1x_1 + 3y_2x_2 + y_3x_3 = \langle \vec{y}, \vec{x} \rangle.$$

iii.

$$\begin{aligned} \langle \alpha\vec{x} + \beta\vec{y}, \vec{z} \rangle &= (\alpha x_1 + \beta y_1)z_1 + 3(\alpha x_2 + \beta y_2)z_2 + (\alpha x_3 + \beta y_3)z_3 \\ &= \dots = \alpha \langle \vec{x}, \vec{z} \rangle + \beta \langle \vec{y}, \vec{z} \rangle \end{aligned}$$

- b. Using this inner product find the projection of $(1, 1, 1)$ onto $(2, -1, 3)$.

$$\begin{aligned} \text{Proj}_{(2,-1,3)}(1, 1, 1) &= \frac{\langle (2, -1, 3), (1, 1, 1) \rangle}{\|(2, -1, 3)\|^2} (2, -1, 3) \\ &= \frac{2}{16} (2, -1, 3) = \frac{1}{8} (2, -1, 3). \end{aligned}$$

- c. Find the projection of $(1, 2, 3)$ onto the plane spanned by $(1, 1, 1)$ and $(2, -1, 3)$.

We need an orthonormal basis for this plane. Set

$$\begin{aligned} \vec{u}_1 &= \frac{(2, -1, 3)}{\|(2, -1, 3)\|} = \frac{(2, -1, 3)}{4} \\ \vec{u}_2 &= \frac{(1, 1, 1) - \frac{1}{8}(2, -1, 3)}{\| \cdot \|} = \frac{\left(\frac{3}{4} \quad \frac{9}{8} \quad \frac{5}{8} \right)}{\sqrt{13}/2} = \frac{(6, 9, 5)}{4\sqrt{19}} \end{aligned}$$

Then the projection of $(1, 2, 3)$ onto this plane (call it P) equals

$$\begin{aligned} \text{Proj}_P(1, 2, 3) &= \text{Proj}_{\vec{u}_1}(1, 2, 3) + \text{Proj}_{\vec{u}_2}(1, 2, 3) \\ &= \frac{5}{4}\vec{u}_1 + \frac{75}{4\sqrt{19}}3\vec{u}_2 = \frac{1}{76}(160, 145, 165) \end{aligned}$$

- d. Find the distance from the point $(1, 2, 3)$ to the plane spanned by $(1, 1, 1)$ and $(2, -1, 3)$.

The distance equals the length of the vector $\vec{v} = (1, 2, 3) - \text{Proj}_P(1, 2, 3)$.

$$\vec{v} = (1, 2, 3) - \frac{1}{76}(160, 145, 165) = \left[\begin{array}{ccc} -\frac{21}{19} & \frac{7}{76} & \frac{63}{76} \end{array} \right]$$

$$\|\vec{v}\| = \frac{7\sqrt{57}}{38}.$$

3. (30) Let L denote the linear transformation that reflects R^3 through the plane $2x - y + z = 0$. Find L 's matrix representation with respect to the standard basis of R^3 .

A basis for this plane is given by $\vec{v}_1 = (1, 0, -2)$ and $\vec{v}_2 = (0, 1, 1)$. Set $\vec{v}_3 = (2, -1, 1)$. These three vectors form a basis for R^3 , and \vec{v}_3 is perpendicular to the plane of reflection. Thus,

$$L(\vec{v}_1) = \vec{v}_1, L(\vec{v}_2) = \vec{v}_2, \text{ and } L(\vec{v}_3) = -\vec{v}_3.$$

Let B be the matrix representation of L with respect to the basis of \vec{v}_i 's. Then

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \text{ Let } P \text{ be the change of basis matrix for which } [\vec{x}]_S = P[\vec{x}]_V.$$

$$\text{That is, } P = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ -2 & 1 & 1 \end{bmatrix}.$$

From the equation $[L(\vec{x})]_S = P[L(\vec{x})]_V = PB[\vec{x}]_V = PBP^{-1}[\vec{x}]_S$

we deduce that the matrix representation, A , of L with respect to the standard basis of R^3 equals PBP^{-1} .

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{5}{6} & \frac{1}{6} \\ \frac{1}{3} & -\frac{1}{6} & \frac{1}{6} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}. \end{aligned}$$

4. (15) Find the straight line $y = mx + b$ that best fits the following data:

$$(1, 1), (2, 1), (3, 2), (4, 7).$$

We want to find m and b such that $y_i = mx_i + b$, where x_i are the entries in the first slots, and y_i are the entries in the second slots. That is, we want to solve the equation

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 7 \end{bmatrix}.$$

If A is the coefficient matrix, the normal equation for the least squares solution to this

problem is $A^T A \begin{bmatrix} m \\ b \end{bmatrix} = A^T \begin{bmatrix} 1 \\ 1 \\ 2 \\ 7 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 37 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 37 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} \frac{19}{10} \\ -2 \end{bmatrix}.$$

Thus, the line of best fit has the equation

$$y = \frac{19}{10}x - 2.$$