1. (25) Let $A = \begin{bmatrix} 1 & 2 & 6 & -4 \\ 1 & 0 & 2 & 6 \\ -1 & 0 & 0 & 5 \end{bmatrix}$.

a. What are the dimensions of the column space and row space of A? What is the dimension of the null space of A?

Since the rows of A are linearly independent, the dimension of the row space is 3, which is therefore also the dimension of the column space of A. The dimension of the null space of A is $4 - 3 = 1$.

b. Find a basis for the null space of A. The matrix A is row equivalent to the matrix

$$\begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -16 \\ 0 & 0 & 1 & 11/2 \end{bmatrix}$$

Thus, the null space of A is spanned by the vector $(10, 32, -11, 2)$

c. Does the equation $A\vec{x} = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}^T$ have a solution?

Since the dimension of the column space of A is 3, which is also the dimension of the co-domain of A, the equation $A\vec{x} = \vec{b}$ is always solvable for any $\vec{b} \in R^3$.

2. (30)

a. Show that $B = \{(1, 6, -1), (4, 0, 1), (0, 1, 1)\}$ is a basis of R^3.

The set B contains 3 vectors, and the dimension of R^3 is also 3. Moreover it is easy to see that B is linearly independent. Thus, it must be a basis of R^3.

b. If $[\vec{x}]_B = [0,1,1]$, then $\vec{x} = ?$

$$\vec{x} = 0(1,6,-1) + (4,0,1) + (0,1,1) = (4,1,2)$$

c. Find the change of basis matrix P such that $[\vec{x}]_S = P[\vec{x}]_B$, where S denotes the standard basis of R^3.

$$P = \begin{bmatrix} 1 & 4 & 0 \\ 6 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

d. If $\vec{x} = (1, -1, 1)$, then $[\vec{x}]_B = ?$

$$[\vec{x}]_B = P^{-1}[\vec{x}]_S = \frac{1}{29}\begin{bmatrix} 1 & 4 & -4 \\ 7 & -1 & 1 \\ -6 & 5 & 24 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{29}\begin{bmatrix} -7 \\ 9 \\ 13 \end{bmatrix}$$

3. (20) Let V and W be two vector spaces and L_1 and L_2 two linear transformations from V to W. Let $B = \{\vec{v}_i\}_{i=1}^{n}$ be a basis for V.

a. Suppose $L_1(\vec{v}_i) = L_2(\vec{v}_i)$ for each $\vec{v}_i \in B$. Show $L_1(\vec{v}) = L_2(\vec{v})$ for every vector $\vec{v} \in V$.

Let $\vec{x}$ be any vector in V. Let $\{\vec{x}_i\}$ be the coordinates of $\vec{x}$ with respect to the given basis. Then we have

$$L_1(\vec{x}) = L_1\left(\sum_{i=1}^{n} x_i\vec{v}_i\right) = \sum_{i=1}^{n} x_i L_1(\vec{v}_i)$$
$$= \sum_{i=1}^{n} x_i L_2(\vec{v}_i) = L_2\left(\sum_{i=1}^{n} x_i\vec{v}_i\right) = L_2(\vec{x})$$

b. True or false: if $\ker(L_1) = \ker(L_2)$, then $\dim(Rg(L_1)) = \dim(Rg(L_2))$. Supply a proof if true and a counter example if false.

The statement is true.

$$\dim(Rg(L_1)) = \dim(V) - \dim(ker(L_1)) = \dim(V) - \dim(ker(L_2)) = \dim(Rg(L_2))$$

4. (30) Define the following quadratic form on R^2

$$Q(x,y) = 22x^2 - 12xy + 13y^2 .$$

a. Find a matrix A such that $Q(x,y) = (x,y) \cdot A \cdot (x,y)^T$.

$$A = \begin{bmatrix} 22 & -6 \\ -6 & 13 \end{bmatrix}$$

b. Plot the locus of points (x,y) such that $Q(x,y) = 1$.

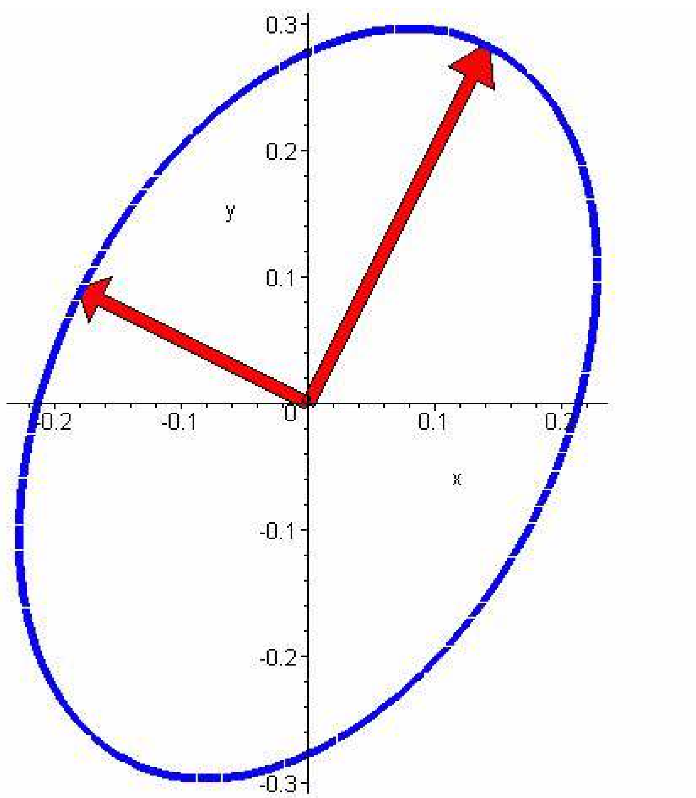
The eigenvalues and eigenvectors of A are

λ	10	25
$\vec{x}_\lambda$	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$	$\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Let $\vec{u}_1 = \frac{1}{\sqrt{5}}(1,2)^T$ and $\vec{u}_2 = \frac{1}{\sqrt{5}}(-2,1)^T$. Let $\vec{x} = x_1\vec{u}_1 + x_2\vec{u}_2$. Then

$$Q(\vec{x}) = (x_1\vec{u}_1 + x_2\vec{u}_2)^T A (x_1\vec{u}_1 + x_2\vec{u}_2)$$
$$= 10x_1^2 + 25x_2^2 = 1.$$

Thus, the curve $Q(x,y) = 1$ is an ellipse with major axis parallel to $\vec{u}_1$ and minor axis parallel to $\vec{u}_2$. A plot of this curve is shown below.



The vectors are parallel to the eigenvectors of the matrix A.

5. (45) Let $A = \begin{bmatrix} 3/4 & -1/4 \\ -1/4 & 3/4 \end{bmatrix}$.

a. Find a diagonal matrix D and a matrix P such that $A = PDP^{-1}$.

The eigenvalues and eigenvectors of the matrix A are

λ	$1/2$	1
$\vec{x}_\lambda$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

.

The matrices D and P are respectively

$$D = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

b. Find a matrix B such that $B^3 = A$.

Since A is similar to a diagonal matrix, taking a cube root of A is easy. All we need to do is take a cube root of D. Thus,

$$\begin{aligned} A^{1/3} &= PD^{1/3}P^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} (1/2)^{1/3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \\ &\approx \begin{bmatrix} 0.896 & -0.103 \\ -0.103 & 0.896 \end{bmatrix} \end{aligned}$$

c. If $\vec{x} = (1, 2)$, find $\lim_{n\to\infty} A^n \vec{x}$

$$\begin{aligned} \lim_{n\to\infty} A^n \begin{pmatrix} 1 \\ 2 \end{pmatrix} &= \lim_{n\to\infty} A^n \left[\frac{3}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] \\ &= \lim_{n\to\infty} \left[\frac{3}{2}\left(\frac{1}{2}\right)^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}1^n \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] \\ &= -\frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}$$