Exercises Chapter VI.

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1. Calculate the lengths of the following vectors:

a. $||(1,2)|| = \sqrt{5}$. b. $||(-1,3,6)|| = \sqrt{1+9+36} = \sqrt{46}$

c. $||(1,1,2,8)|| = \sqrt{1+1+4+84} = \sqrt{90}$.

2. Find all unit vectors that are parallel to the vector $(1,2,-4)$.

If $\boldsymbol{x} = (x_1, x_2, x_3)$ is parallel to $(1,2,-4)$, then $(x_1, x_2, x_3) = \lambda(1,2,-4)$. The length of this vector is $|\lambda|\sqrt{21}$. If $\boldsymbol{x}$ is to have unit length then $\lambda = \pm 1/\sqrt{21}$. Thus,

$$\boldsymbol{x} = \pm \frac{(1,2,-4)}{\sqrt{21}}\,.$$

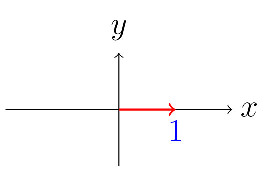
3. Compute the dot product of each of the following pairs of vectors:

a. $< (1,0),(0,1) >= 0$. b. $< (a,b),(b,a) >= 2ab$

c. $< (1,2,1),(3,-6,2) >= 3 - 12 + 2 = -7$

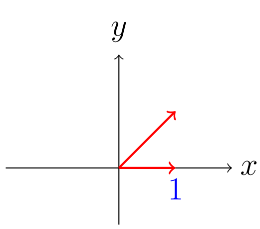
4. Sketch each of the following pairs of vectors. Compute their inner product and determine the cosine of the angle between them.

a. (1,0), (1,0)



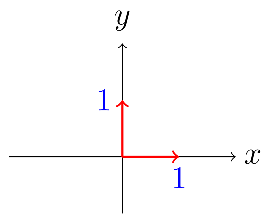
$< (1,0),(1,0) >= 1$, and $\cos\theta = \frac{1}{1\cdot 1} = 1$.

b. (1,0), (1,1)



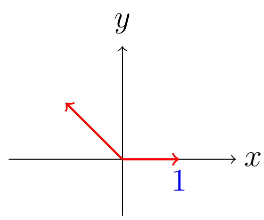
$< (1,0),(1,1) >= 1$ and $\cos\theta = \frac{1}{1\cdot\sqrt{2}} = \frac{1}{\sqrt{2}}$.

c. (1,0), (0,1)



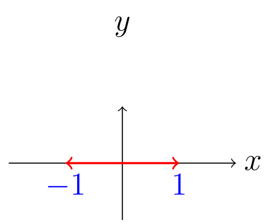
$< (1,0),(0,1) >= 0$ and $\cos\theta = 0\,.$

d. (1,0), $(-1,1)$



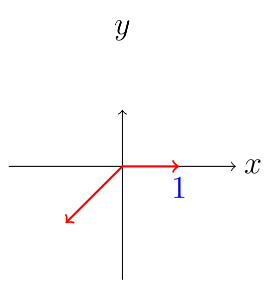
$< (1,0),(-1,1) >= -1$ and $\cos\theta = \frac{-1}{\sqrt{2}}\,.$

e. (1,0), $(-1,0)$



$< (1,0),(-1,0) >= -1$ and $\cos\theta = -1.$

f. (1,0), $(-1,-1)$



$< (1,0),(-1,-1) >= -1$ and $\cos\theta = \frac{-1}{\sqrt{2}}.$

5. Find the cosine of the angle between each of the following pairs of vectors:

a. (1,2), (3,-1), $\cos\theta = \dfrac{1}{\sqrt{5}\sqrt{10}} = \dfrac{1}{5\sqrt{2}}.$

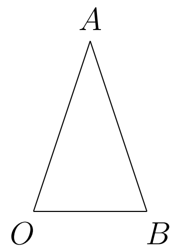
b. $(1, 0, -4)$, (6,1,2), $\cos\theta = \dfrac{-2}{\sqrt{17}\sqrt{41}}.$

c. $(-2, 3, 0, 1), (1, 2, 8, -2)$, $\cos\theta = \dfrac{-2}{\sqrt{14}\sqrt{73}}.$

16. Suppose that $\boldsymbol{x}$ is perpendicular to every vector in some set A. Show that $\boldsymbol{x}$ must then be perpendicular to every vector in $S[A]$.

Suppose a vector $\boldsymbol{y}$ is in $S[A]$. Then there are vectors $\{\boldsymbol{a}_i\}_{i=1}^k$ in A, and constants c_i such that $\boldsymbol{y} = \sum_{i=1}^k c_i\boldsymbol{a}_i$. Using the linearity of the inner product we have

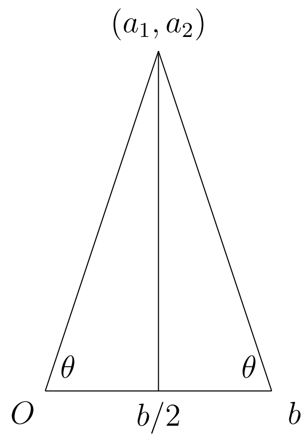
$$< \boldsymbol{x}, \boldsymbol{y} >=< \boldsymbol{x}, \sum_{i=1}^{k} c_i\boldsymbol{a}_i >= \sum_{i=1}^{k} c_i < \boldsymbol{x}, \boldsymbol{a}_i >= 0\,.$$

19. Let 

be an isosceles triangle with equal angles at O and B.

Show that the line drawn from the vertex A to the midpoint of OB is perpendicular to OB.

Assume that the point O is at the origin $(0,0)$. Draw the line from point A to the midpoint of line $\overline{OB}$. Let θ denote the value of the equal angles, and suppose A has coordinates (a_1, a_2), the midpoint has coordinates $(b/2, 0)$, and B has coordinates $(b, 0)$, with $a_1 > 0,\ a_2 > 0$, and $b > 0$. See the figure below.



We have the following expressions for the cosine of the angle θ:

$$\begin{aligned} \cos\theta = \frac{< (a_1, a_2), (b/2, 0) >}{\sqrt{a_1^2 + a_2^2}(b/2)} &= \frac{< (a_1 - b, a_2), (-b/2, 0) >}{\sqrt{(a_1 - b)^2 + a_2^2}(b/2)} \\ \frac{a_1}{\sqrt{a_1^2 + a_2^2}} &= \frac{(b - a_1)}{\sqrt{(a_1 - b)^2 + a_2^2}} \\ 0 &= a_2^2 b(b - 2a_1) \end{aligned}$$

Since $a_2^2 b \neq 0$, we have $a_1 = b/2$. Thus, the vertex A sits directly about the midpoint B. From this it is clear that the line from A to the midpoint is perpendicular to the base of the triangle.

1. Compute $\text{Proj}_u \boldsymbol{x}$, where $\boldsymbol{x} = (7, -8)$ for each of the following unit vectors:

a. $(1, -2)/5^{1/2}$: $\text{Proj}_u \boldsymbol{x} = \frac{23}{5}(1, -2)$.

b. $(2, 3)/(13)^{1/2}$: $\text{Proj}_u \boldsymbol{x} = \frac{-10}{13}(2, 3)$.

c. $(1, 0)$: $\text{Proj}_u \boldsymbol{x} = 7(1, 0)$.

3. Let $\boldsymbol{x} = (7, -5)$. Let $U = \{(1, 5)/\sqrt{26}, (-5, 1)/\sqrt{26}\}$.

a. Show that U is an orthonormal basis of $\mathbb{R}^2$.
Each vector in U has length 1, and the dot product of $(1, 5)$ with $(-5, 1)$ is zero. Thus, U is an orthonormal basis of $\mathbb{R}^2$.

b. Find $\text{Proj}_{u_j} \boldsymbol{x}$, where $\boldsymbol{u}_j$ is the jth unit vector in U.
The projections of $\boldsymbol{x}$ are:

$$\begin{aligned} \text{Proj}_{\boldsymbol{u}_1} \boldsymbol{x} &= \frac{-18}{\sqrt{26}} \boldsymbol{u}_1 \\ \text{Proj}_{\boldsymbol{u}_2} \boldsymbol{x} &= \frac{-40}{\sqrt{26}} \boldsymbol{u}_2 \end{aligned}$$

c. Compute the coordinates of $\boldsymbol{x}$ with respect to U.
The coordinates of $\boldsymbol{x}$ with respect to the basis U are:

$$[\boldsymbol{x}]_U = \left[\frac{-18}{\sqrt{26}}, \frac{-40}{\sqrt{26}} \right].$$

6. Let $V = \{(2, -3, 1), (2, 3, 5), (-9, -4, 5)\}$.

a. Show that V is an orthogonal set of vectors.
Just computer the various dot products. For future reference the lengths of the vectors in V are computed.

$$||\boldsymbol{v}_1|| = \sqrt{14}, \qquad ||\boldsymbol{v}_2|| = \sqrt{38}, \qquad ||\boldsymbol{v}_3|| = \sqrt{122}.$$

b. Let $\boldsymbol{x} = (7, -3, 4)$. Compute the projection of $\boldsymbol{x}$ onto the direction given by $\boldsymbol{v}_j$, where $\boldsymbol{v}_j$ is the jth vector in the set V.

$$\text{Proj}_{\boldsymbol{v}_1} \boldsymbol{x} = \frac{27}{14} \boldsymbol{v}_1, \qquad \text{Proj}_{\boldsymbol{v}_2} \boldsymbol{x} = \frac{25}{38} \boldsymbol{v}_2, \qquad , \text{Proj}_{\boldsymbol{v}_3} \boldsymbol{x} = \frac{-31}{122} \boldsymbol{v}_3,$$

c. Compute the coordinates of $(7, -3, 4) = \boldsymbol{x}$ with respect to the basis V. (Hint: It's easy to construct an orthonormal basis from V.)

$$[\boldsymbol{x}]_V = \left[\frac{27}{14}, \frac{25}{38}, \frac{-31}{122} \right].$$

7. Find the angle between the following pairs of vectors:

a. (1,1), (0,1) b. (1,1,1), (0,1,0)

c. (1,1,1,1), (0,1,0,0) d. $(6,7,-2,3),(-1,-2,1,1)$

a. $\cos\theta = \frac{1}{\sqrt{2}} \implies \theta = \arccos\frac{1}{\sqrt{2}} = \frac{\pi}{4}$.

b. $\cos\theta = \frac{1}{\sqrt{3}} \implies \theta = \arccos\frac{1}{\sqrt{3}} \approx 0.955\,\text{rad}$.

c. $\cos\theta = \frac{1}{\sqrt{4}} \implies \theta = \arccos\frac{1}{2} = \frac{\pi}{3}$.

d. $\cos\theta = \frac{-19}{\sqrt{98}\sqrt{7}} \implies \theta = \arccos\frac{-19}{\sqrt{686}} \approx 2.382\,\text{rad}$.

10. Let $\boldsymbol{u}$ be an arbitrary unit vector in $\mathbb{R}^n$.

a. If $\boldsymbol{x}$ is the zero vector, show that $\text{Proj}_u\boldsymbol{x} = \boldsymbol{0}$.

$$\text{Proj}_u\boldsymbol{x} =< \boldsymbol{0}, \boldsymbol{u} > \boldsymbol{u} = 0\boldsymbol{u} = \boldsymbol{0}\,.$$

b. If $\boldsymbol{x}$ and $\boldsymbol{u}$ are perpendicular, show that $\text{Proj}_u\boldsymbol{x} = \boldsymbol{0}$.

$$\text{Proj}_u\boldsymbol{x} =< \boldsymbol{x}, \boldsymbol{u} > \boldsymbol{u} = 0\boldsymbol{u} = \boldsymbol{0}\,.$$

c. Show that $\text{Proj}_u\boldsymbol{x}$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$.

$$\begin{aligned}\text{Proj}_u(\boldsymbol{x}+\boldsymbol{y}) &= < \boldsymbol{x}+\boldsymbol{y}, \boldsymbol{u} > \boldsymbol{u} =< \boldsymbol{x}, \boldsymbol{u} > \boldsymbol{u}+ < \boldsymbol{y}, \boldsymbol{u} > \boldsymbol{u} = \text{Proj}_u\boldsymbol{x} + \text{Proj}_u\boldsymbol{y} \\ \text{Proj}_u(\alpha\boldsymbol{x}) &= < \alpha\boldsymbol{x}, \boldsymbol{u} > \boldsymbol{u} = \alpha < \boldsymbol{x}, \boldsymbol{u} > \boldsymbol{u} = \alpha\text{Proj}_u\boldsymbol{x}\end{aligned}$$

d. What is the dimension of the kernel of this linear transformation?
Since the dimension of the range of the projection mapping is one, the kernel must have dimension $n-1$.

11. Let $V = P_3$. Let $\boldsymbol{f}$ and $\boldsymbol{g}$ be any two polynomials in V. Define $\langle \boldsymbol{f}, \boldsymbol{g}\rangle = \int_0^1 \boldsymbol{f}(t)\boldsymbol{g}(t)dt$; cf. problem 8 in Section 6.1.

a. Find a unit vector $\boldsymbol{u}$ that points in the same direction as $\boldsymbol{f}(t) = t$.

$$\boldsymbol{u} = \frac{1}{||\boldsymbol{f}||}\boldsymbol{f} = \frac{1}{\left(\int_0^1 t^2\,dt\right)^{1/2}}t = \sqrt{3}\,t\,.$$

b. Find the projection of $\boldsymbol{t}^2$ onto the vector $\boldsymbol{u}$ of part a.

$$\text{Proj}_{\boldsymbol{u}}\boldsymbol{t}^2 \;=\; < t^2, \boldsymbol{u} > \boldsymbol{u} = \left(\int_0^1 t^2(\sqrt{3}\,t)\,dt\right)\sqrt{3}t = \frac{3}{4}t\,.$$

c. Find the cosine of the angle between the vectors $\boldsymbol{t}^2$ and $\boldsymbol{t}$.

$$\cos\theta = \frac{< t, t^2 >}{||t||\,||t^2||} = \frac{\int_0^1 t^3\,dt}{\left(\int_0^1 t^2\,dt\right)^{1/2}\left(\int_0^1 t^4\,dt\right)^{1/2}} = \frac{1/4}{\sqrt{1/3}\sqrt{1/5}} = \frac{\sqrt{15}}{4}$$

13. Let $V = P_2$. Define the inner product as we did in problems 11 and 12. Let $\boldsymbol{f}'$ denote the derivative of $\boldsymbol{f}$.

a. Find all polynomials in P_2 that are perpendicular to their derivatives.
The inner product of f with its derivative is

$$
\begin{aligned}
< \boldsymbol{f}, \boldsymbol{f}' > &= \int_0^1 f(t)\, f'(t)\, dt = \frac{1}{2}\int_0^1 (\frac{d}{dt} f^2)\, dt = \frac{1}{2}\left(f^2(1) - f^2(0)\right) \\
&= \frac{1}{2}\left(f(1) - f(0)\right)\left(f(1) + f(0)\right)
\end{aligned}
$$

Thus, $\boldsymbol{f}$ is orthogonal to its derivative if

$$f(1) - f(0) = 0 \quad \text{or} \quad f(1) + f(0) = 0$$

Note that this is a non-linear condition. So the subset of such polynomials will not necessarily be a subspace. The general polynomial in P_2 has the form $a_0 + a_1 t + a_2 t^2$. The two equations above say that $\boldsymbol{p}$ must have one of the following two forms:

$$p(t) = a_0 + a_1 t - a_1 t^2, \quad \text{or} \quad p(t) = a_o + a_1 t - (2a_0 + a_1) t^2$$

b. For any two polynomials f and g in V, compute $\langle \boldsymbol{f}, \boldsymbol{g}' \rangle + \langle \boldsymbol{f}', \boldsymbol{g} \rangle$.

$$
\begin{aligned}
\langle \boldsymbol{f}, \boldsymbol{g}' \rangle + \langle \boldsymbol{f}', \boldsymbol{g} \rangle &= \int_0^1 f(t) g'(t)\, dt + \int_0^1 f'(t) g(t)\, dt = \int_0^1 \frac{d}{dt}\left(f(t)g(t)\right)'\, dt \\
&= f(t)g(t)|_0^1 = f(1)g(1) - f(0)g(0)
\end{aligned}
$$

1. Use the Gram–Schmidt procedure to construct an orthonormal basis for each of the following subspaces of $\mathbb{R}^3$:

a. $W = \{(x_1, x_2, x_3)\colon\ x_1 - x_2 = 0\}$
One basis for W is $\{(1,1,0),(0,0,1)\}$. Using Gram-Schmidt we have

$$\begin{aligned} \boldsymbol{u}_1 &= \frac{1}{\sqrt{2}}(1,1,0) \quad \boldsymbol{v}_1 = (0,0,1) - \text{Proj}_{\boldsymbol{u}_1}(0,0,1) = (0,0,1) \\ \boldsymbol{u}_2 &= \frac{\boldsymbol{v}_1}{||\boldsymbol{v}_1||} = (0,0,1) \end{aligned}$$

Thus, an orthonormal basis of W is $\left\{\frac{1}{\sqrt{2}}(1,1,0),(0,0,1)\right\}$.

b. $W = S[(1,-1,2),(6,1,1)]$
Since the vectors used to generate (span) W are linearly independent they form a basis of W.

$$\begin{aligned} \boldsymbol{u}_1 &= \frac{1}{\sqrt{6}}(1,-1,2) \\ \boldsymbol{v}_1 &= (6,1,1) - \text{Proj}_{\boldsymbol{u}_1}(6,1,1) = (6,1,1) - \frac{7}{6}(1,-1,2) = \frac{1}{6}(29,13,-8) \\ \boldsymbol{u}_2 &= \frac{1}{\sqrt{234}}(29,13,-8) \end{aligned}$$

An orthonormal basis of W is $\left\{\frac{1}{\sqrt{6}}(1,-1,2), \frac{1}{\sqrt{1074}}(29,13,-8))\right\}$.

2. Construct an orthonormal basis for $\mathbb{R}^3$ from the following basis,

$\{(0,5,1),\ (0,1,-5),\ (1,-2,3)\}$.

$$\begin{aligned} \boldsymbol{u}_1 &= \frac{(0,5,1)}{\sqrt{26}}, \quad \boldsymbol{u}_2 = \frac{(0,1,-5)}{\sqrt{26}} \\ \boldsymbol{v}_3 &= \boldsymbol{v}_3 - \text{Proj}_{\boldsymbol{u}_1}\boldsymbol{v}_3 - \text{Proj}_{\boldsymbol{u}_2}\boldsymbol{v}_3 = (1,-2,3) - \left(\frac{-7}{26}(0,5,1)\right) - \left(\frac{-17}{26}(0,1,-5)\right) \\ &= (1,0,0). \end{aligned}$$

Thus, an orthonormal basis is $\left\{\frac{(0,5,1)}{\sqrt{26}},\ \frac{(0,1,-5)}{\sqrt{26}},\ (1,0,0)\right\}$.

4. Find the distance from the point $(1, -2, 3)$ to the plane $2x_1 - 3x_2 + 6x_3 = 0$.

A direction normal to the plane is given by the vector $\boldsymbol{n} = (2, -3, 6)$, and $(3, 2, 0)$ is a point on the plane. The vector

$$\boldsymbol{x} = (1, -2, 3) - (3, 2, 0) = (-2, -4, 3)$$

can be thought of as starting at a point on the plane and terminating at the point $(1, -2, 3)$. Thus, the projection of $\boldsymbol{x}$ onto $\boldsymbol{n}$ can be thought of as a vector starting on the plane, which is perpendicular to the plane, and terminates at the given point. So the length of $\boldsymbol{x}$ will give us the distance from the point to the plane.

$$\begin{aligned} \text{Proj}_{\boldsymbol{n}}\boldsymbol{x} &= \frac{-4 + 12 + 18}{4 + 16 + 9}(2, -3, 6) = \frac{10}{29}(2, -3, 6) \\ \text{distance to plane} &= \left\| \frac{10}{29}(2, -3, 6) \right\| = \frac{70}{29}. \end{aligned}$$

5. Find the distance from the point $(1, -2, 3)$ to the plane $2x_1 - 3x_2 + 6x_3 = 2$.

The computation is exactly the same as in the previous problem. Set $\boldsymbol{n} = (2, -3, 6)$ and $\boldsymbol{x} = (1, -2, 3) - (1, 0, 0) = (0, -2, 3)$.

$$\begin{aligned} \text{Proj}_{\boldsymbol{n}}\boldsymbol{x} &= \frac{24}{29}(2, -3, 6) \\ \text{distance to plane} &= \left\| \frac{24}{29}(2, -3, 6) \right\| = \frac{168}{29} \end{aligned}$$

9. Find an orthonormal basis for the kernels of each of the following matrices:

a. $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ b. $\begin{bmatrix} 1 & -1 & 2 \\ 4 & 6 & 3 \end{bmatrix}$ c. $\begin{bmatrix} 1 & 0 & -1 & 3 \\ -3 & 1 & 0 & 1 \end{bmatrix}$

a. Matrix has rank equal to 2, so kernel has dimension 1. A basis is $\{(2, -1)\}$, and an orthonormal basis is $\left\{ \frac{(2,-1)}{\sqrt{5}} \right\}$.

b. Matrix has rank equal to 2, so kernel has dimension 1. A basis is $\{(-3/2, 1/2, 1)\}$, and an orthonormal basis is $\left\{ \frac{-3,1,2)}{\sqrt{14}} \right\}$.

c. Matrix has rank equal to 2, so kernel has dimension 2. A basis is $\{(1, 3, 1, 0), (-3, -10, 0, 1)\}$. Using the Gram-Schmidt procedure we construct an orthonormal basis $\left\{ \frac{(1,3,1,0)}{\sqrt{11}}, \frac{(0,-1,3,1)}{\sqrt{11}} \right\}$.

10. Find an orthonormal basis for the ranges of each of the matrices in problem 9.

a. The range of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ equals S$[(1,3)]$. An orthonormal basis is $\{\frac{(1,3)}{\sqrt{10}}\}$.

b. and c. The ranges of the matrices in parts b. and c. are equal to $\mathbb{R}^2$. Thus, an orthonormal basis for their ranges is $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$.

13. Let $V = P_2$. Define $\langle \boldsymbol{f}, \boldsymbol{g} \rangle = \int_0^1 \boldsymbol{f}(t)\boldsymbol{g}(t)dt$. The set $B = \{1, t, t^2\}$ is a basis for V. Construct an orthonormal basis for V from B by using the Gram–Schmidt procedure.

$$\boldsymbol{u}_1 = \frac{\mathbf{1}}{\left(\int_0^1 1^2\, dt\right)^{1/2}} = \mathbf{1}$$

$$\boldsymbol{v}_2 = t - \text{Proj}_{\boldsymbol{u}_1} t = t - \frac{1}{2}$$

$$\boldsymbol{u}_2 = \frac{\boldsymbol{v}_2}{\left(\int_0^1 (t-1/2)^2\, dt\right)^{1/2}} = \sqrt{12}\left(t - \frac{1}{2}\right)$$

$$\boldsymbol{v}_3 = t^2 - \text{Proj}_{\boldsymbol{u}_1} t^2 - \text{Proj}_{\boldsymbol{u}_2} t^2 = t^2 - t + \frac{1}{6}$$

$$\boldsymbol{u}_3 = \frac{\boldsymbol{v}_3}{\left(\int_0^1 (t^2 - t + \frac{1}{6})^2\, dt\right)^{1/2}} = 6\sqrt{5}\left(t^2 - t + \frac{1}{6}\right)$$

16. Let $V = S[(1,0,1),(1,1,1)]$. Show that $(-1,0,1)$ is perpendicular to every vector in V.

Let $\boldsymbol{x}_1 = (1,0,1)$ and $\boldsymbol{x}_2 = (1,1,1)$. We first observe that the vector $(-1,0,1)$ is perpendicular to both of the $\boldsymbol{x}'s$.

$$\langle (-1,0,1), \boldsymbol{x}_1 \rangle = -1 + 1 = 0 \quad \langle (-1,0,1), \boldsymbol{x}_2 \rangle = -1 + 1 = 0\,.$$

If $\boldsymbol{x} \in V$, then there are constants c_1 and c_2 such that $\boldsymbol{x} = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2$. Thus,

$$\begin{aligned} \langle (-1,0,1), \boldsymbol{x} \rangle &= \langle (-1,0,1), c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 \rangle = c_1\langle (-1,0,1), \boldsymbol{x}_1 \rangle + c_2\langle (-1,0,1), \boldsymbol{x}_2 \rangle \\ &= 0 + 0 = 0 \end{aligned}$$

5. Find an orthonormal basis of eigenvectors for each of the following matrices:

a. $\begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}$.

The eigenvalues are $\lambda = -1 \pm 2\sqrt{2}$. Eigenvectors associated with them are: $(1, -1+\sqrt{2})$ and $(1, -1-\sqrt{2})$ respectively. These eigenvectors are orthogonal, so to get an orthonormal basis we just need to normalize them.

$$\boldsymbol{u}_1 = \frac{(1, -1+\sqrt{2})}{\sqrt{4-2\sqrt{2}}} \qquad \boldsymbol{u}_2 = \frac{(1, -1-\sqrt{2})}{\sqrt{4+2\sqrt{2}}}$$

b. $\begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$

Eigenvalues and eigenvectors are: 6 and 4, with $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, respectively. The eigenvectors $\boldsymbol{e}_i$ are an orthonormal basis of $\mathbb{R}^2$.

c. $\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$

The eigenvalues and their associated eigenvectors are: $\frac{5+\sqrt{5}}{2}, (-2, -1+\sqrt{5})$ and $\frac{5-\sqrt{5}}{2}, (-2, -1-\sqrt{5})$. The eigenvectors are orthogonal so we just need to normalize them.

$$\boldsymbol{u}_1 = \frac{(-2, -1+\sqrt{5})}{\sqrt{10-2\sqrt{5}}} \quad \boldsymbol{u}_2 = \frac{(-2, -1-\sqrt{5})}{\sqrt{10+2\sqrt{5}}}$$

6. Find an orthonormal basis of eigenvectors for each of the following matrices:

a. $\begin{bmatrix} 8 & -1 & 1 \\ -1 & 8 & 1 \\ 1 & 1 & 8 \end{bmatrix}$

The eigenvalues and eigenvectors are:

λ	$\boldsymbol{x}_\lambda$
6	$(-1, -1, 1)$
9	$(1, 0, 1), (-1, 1, 0)$

Notice that the eigenvector associated with 6 is orthogonal to the two eigenvectors associated with the eigenvalue 9. Using the Gram-Schmidt algorithm we construct from this basis of eigenvectors an orthonormal basis of eigenvectors.

$$\boldsymbol{u}_1 = \frac{(-1, -1, 1)}{\sqrt{3}}, \quad \boldsymbol{u}_2 = \frac{(1, 0, 1)}{\sqrt{2}}, \quad \boldsymbol{u}_3 = \frac{(-1, 2, 1)}{\sqrt{6}}$$

b. $\begin{bmatrix} -2 & 3 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

The eigenvalues and eigenvectors are:

λ	$\boldsymbol{x}_\lambda$
2	$(0,0,1)$
$1+3\sqrt{2}$	$(1,1+\sqrt{2},0)$
$1-3\sqrt{2}$	$(1,1-\sqrt{2},0)$

An orthogonal basis is:

$$\boldsymbol{u}_1 = (0,0,1), \quad \boldsymbol{u}_2 = \frac{(1,1+\sqrt{2},0)}{\sqrt{4+2\sqrt{2}}}, \quad \boldsymbol{u}_3 = \frac{(1,1-\sqrt{2},0)}{\sqrt{4-2\sqrt{2}}}$$

8. For each of the matrices A of problem 5 find P and D such that $P^TAP = D$, where D is a diagonal matrix.

a. $\begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}$.

The eigenvalues are: $\lambda = -1 \pm 2\sqrt{2}$, and an orthonormal basis of eigenvectors of this matrix is $\left\{ \frac{(1,-1+\sqrt{2})}{\sqrt{4-2\sqrt{2}}}, \frac{(1,-1-\sqrt{2})}{\sqrt{4+2\sqrt{2}}} \right\}$ respectively. Thus, we have

$$D = \begin{bmatrix} -1+2\sqrt{2} & 0 \\ 0 & -1-2\sqrt{2} \end{bmatrix}, \quad P = \begin{bmatrix} \frac{1}{\sqrt{4-2\sqrt{2}}} & \frac{1}{\sqrt{4+2\sqrt{2}}} \\ \frac{-1+2\sqrt{2}}{\sqrt{4-2\sqrt{2}}} & \frac{-1-2\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \end{bmatrix}.$$

b. $\begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$ This matrix is already diagonal. So set D equal to it, and $P = I_2$.

c. $\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$. An orthornormal basis of eigenvectors of A is:

$$\boldsymbol{u}_1 = \frac{(-2,-1+\sqrt{5})}{\sqrt{10-2\sqrt{5}}} \quad \boldsymbol{u}_2 = \frac{(-2,-1-\sqrt{5})}{\sqrt{10+2\sqrt{5}}},$$

with eigenvalues $\frac{5+\sqrt{5}}{2}$ and $\frac{5-\sqrt{5}}{2}$ respectively. Thus,

$$D = \begin{bmatrix} \frac{5+\sqrt{5}}{2} & 0 \\ 0 & \frac{5-\sqrt{5}}{2} \end{bmatrix}, \text{ and } P = \begin{bmatrix} \frac{-2}{\sqrt{10-2\sqrt{5}}} & \frac{-2}{\sqrt{10+2\sqrt{5}}} \\ \frac{-1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}} & \frac{-1-\sqrt{5}}{\sqrt{10+2\sqrt{5}}} \end{bmatrix}$$

9. For each of the matrices A of problem 6 find P and D such that $P^T AP = D$, where D is a diagonal matrix.

a. $\begin{bmatrix} 8 & -1 & 1 \\ -1 & 8 & 1 \\ 1 & 1 & 8 \end{bmatrix}$

The eigenvalues and an orthonormal basis of eigenvectors are given in the table below.

λ	$\boldsymbol{x}_\lambda$
6	$\frac{(-1,-1,1)}{\sqrt{3}}$
9	$\frac{(1,0,1)}{\sqrt{2}}, \frac{(-1,2,1)}{\sqrt{6}}$

Thus, the matrices are:

$$D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}, \qquad P = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

1. Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 3 & 4 \end{bmatrix}$

a. Determine the range of A, and show that (1,1,0) is not in the range, i.e., the equation $A\boldsymbol{x} = (1,1,0)^T$ does not have a solution.

The range of A is the span of the two columns of A. To see that $\boldsymbol{x} = (1,1,0)^T$ is not in the range of A, note that the augmented matrix of this system has rank 3. That is, the rank of A and the rank of the augmented matrix are not equal, which means that the equation $A\boldsymbol{x} = (1,1,0)^T$ does not have a solution.

b. Compute A^TA and show that it is one to one.

$$A^TA = \begin{bmatrix} 2 & 0 & 3 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 13 & 14 \\ 14 & 21 \end{bmatrix}, \quad \det\left(A^TA\right) = 77\,.$$

Since the determinant of A^TA is not zero, this matrix is one-to-one.

c. Solve the equation $A^TA\boldsymbol{x} = A^T\boldsymbol{b}$, where $\boldsymbol{b} = (1,1,0)$.

$$A^TA\boldsymbol{x} \quad = \quad A^T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

The solution to this system of equations is: $\boldsymbol{x} = (0, 1/7)$.

d. If $\boldsymbol{x}$ is your solution from part c, show that $\|A\boldsymbol{x} - \boldsymbol{b}\|$ is smaller than $\|\boldsymbol{w} - \boldsymbol{b}\|$ for any vector $\boldsymbol{w}$ in the range of A.
We know that $\|\text{Proj}_{Rg(A)}\boldsymbol{b} - \boldsymbol{b}\| \leq \|\boldsymbol{w} - \boldsymbol{b}\|$ for all $\boldsymbol{w} \in Rg(A)$, and since $A\boldsymbol{x} = \text{Proj}_{Rg(A)}\boldsymbol{b}$, we have the desired inequality.

2. Determine the straight-line least squares fit for the following data: (1,1), $(2,-3)$, (4,0), (5,1), (10,3).

We want to find m and b such that the data satisfies the equation $y = mx + b$. This leads to the following system of equations for the unknowns m and b:

$$m + b = 1, \quad 2m + b = -3, \quad 4m + b = 0, \quad 5m + b = 1, \quad 10m + b = 3\,.$$

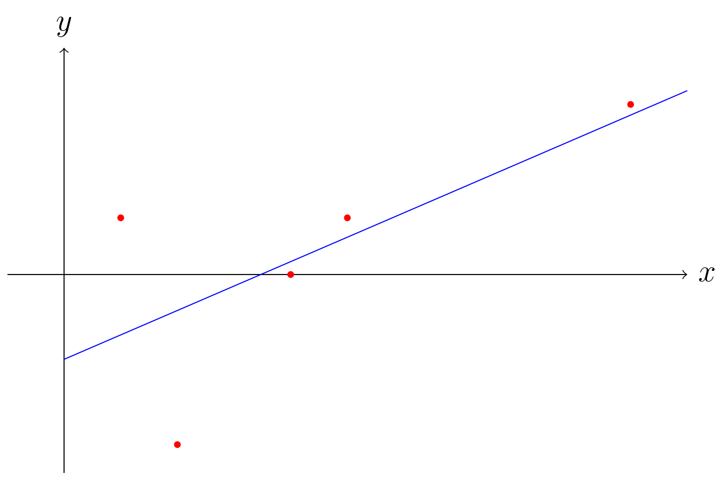
Writing this as a matrix equation we have

$$A\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \\ 10 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \\ 3 \end{bmatrix}.$$

Note that A has full rank, i.e., 2, which means that A^TA is invertible. And the best possible values of m and b are given as solutions of

$$A^TA\begin{bmatrix} m \\ b \end{bmatrix} = A^T\begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 & 5 & 10 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 30 \\ 2 \end{bmatrix}.$$

That is, $\begin{bmatrix} 146 & 22 \\ 22 & 5 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 30 \\ 2 \end{bmatrix}$, and $m = 53/123$, $b = -184/123$. A plot of the data points and the straight line least squares fit is shown below.



4. Consider the system of equations:

$$\begin{aligned} 3x_1 + 4x_2 + 8x_3 &= 0 \\ x_1 \qquad\quad - \ x_3 &= 1 \\ 2x_1 + \ x_2 + 4x_3 &= 0 \\ x_1 + \ x_2 + \ x_3 &= 0 \end{aligned}$$

a. This system is overdetermined (more equations than unknowns) and may not have a solution. Show that if there is a solution, it is unique.

The coefficient matrix

$$A = \begin{bmatrix} 3 & 4 & 8 \\ 1 & 0 & -1 \\ 2 & 1 & 4 \\ 1 & 1 & 1 \end{bmatrix}$$

has rank equal to 3. Thus, its null space is just the zero vector and hence A defines a 1-to-1 linear transformation.

b. Show that this system does not have a solution, and then find $\boldsymbol{x}$ in $\mathbb{R}^3$ such that $A\boldsymbol{x}$ is that vector in the range of A closest to (0,1,0,0).

The augmented matrix of this system has rank equal to 4, which does not equal the rank of the coefficient matrix. Thus, the system does not have a solution. To find $\boldsymbol{x}$, we only have to solve the normal equations:

$$A^T A\boldsymbol{x} = A^T \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 15 & 15 & 32 \\ 15 & 18 & 37 \\ 32 & 37 & 82 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Solving this system we get $x_1 = 128/243, x_2 = 29/243, x_3 = -22/81$.

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3. Compute the inner product and the cosine of the angle between each of the following pairs of vectors:

a. $(-4, 5)$, (1,2)

$$\langle(-4,5),(1,2)\rangle = -4+10 = 6, \quad \cos\theta = \frac{6}{\sqrt{205}}.$$

b. $(-2, 3, 7), (2, -4, 5)$

$$\langle(-2,3,7),(2,-4,5)\rangle = -4-12+35 = 19, \quad \cos\theta = \frac{19}{\sqrt{62}\sqrt{45}}.$$

c. $(-1, -2, 3, 5)$, (1,1,0,8)

$$\langle(-1,-2,3,5),(1,1,0,8)\rangle = -1-2+40 = 37, \quad \cos\theta = \frac{37}{\sqrt{39}\sqrt{66}}.$$