Following each question is the answer to that question. The details of the calculations can be found in the Maple worksheet which is at the end of this solution set.

Bond	Coupon Rate	Matures	Asked
B_1	5.5	December 21, 2001	101.50
B_2	5.0	August 8, 2002	103.00

1. (30) Determine the invoice prices and yields to maturity for the two bonds in the table above. The asked prices are flat price quotes for a \$100 face value bond. Assume the face value for each bond is \$100.00, and that the settlement date is November 29, 2001.

   The accrued interest for bonds B_1 and B_2, their invoice prices, and yields are:

$$\begin{aligned} AI_1 &= 2.419 \qquad P_1 = 103.92 \\ AI_2 &= 1.535 \qquad P_2 = 104.53 \\ y_1 &= -18.72 \qquad y_2 = 0.652 \end{aligned}$$

2. (10) Determine the Macaulay duration for bond B_2 in the table above.

   duration equals 1.3726

3. (10) Assume that immediately after buying bond B_2 there is a decrease of 10 basis points in its yield Using the Macaulay duration you found in question 2 estimate the new flat price of the bond.

   The new flat price is

   103.07

4. (20) The table below contains data on two bonds. The first bond matures in one interest period and the second bond in two interest periods.

Bond	Coupon Rate	Yield	Time to Maturity in half years
B_1	5.5	5	1
B_2	5.0	4	2

Suppose you can buy or sell a CD which matures in 1 year at an interest rate of 3.5% per year. That is, after 1 year owning a \$100 CD will pay you \$103.50. Assume that you can buy or sell either of the two bonds at the given yields, in any dollar amounts.

(a) Explain why this presents an arbitrage opportunity. Hint: What is the time t_0 interest rate structure?

The time t_0 interest rate structure which is implied by the yields of these two bonds is $r(0,1) = 5.5$ and $r(0,2) = 3.988$. Since $r(0,2)$ is larger than the interest rate of a one year CD an arbitrage opportunity presents itself.

(b) Explain how to use this opportunity to make free money.

Sell \$100.97 worth of the CD and buy one bond B_2. The \$100.97 is the current invoice price of B_2. After one year you will owe $100.97(1.03) = 104.00$. The money earned when the bond matures is 105 dollars, and this does not include any interest the first coupon payment could earn. Thus, there is a guarantied risk-free profit of \$1.00

5. (10) A 50,000 treasury note which will mature in 20 years has a coupon rate of 4% and a yield to maturity of 3.75%. In the belief that interest rates are going to rise an investor short sells the bond. Suppose at the start of the next coupon period the yield has increased to 3.80%, and the investor closes out the position. Assuming a reinvestment rate of 4% determine the net profit or loss to the investor.

The current price of the bond is \$51,747.80 and its price after 6 months, with the increase in yield taken into account is \$51,368.50. By investing the 51,747.80 for 6 months at 4% the investor has available 52,782.75 from which the coupon payment and the cost of buying back the bond must be taken. This leaves a net profit of

$$52,782.75 - 52,258.50 = 414.25$$

6. (10) The futures price, f, of a futures contract is given by the formula

$$f = P_0\left(1 + \frac{i - y_c}{100}t\right),$$

where t is the time in years until the contract matures, y_c is the current yield of the bond, i is the borrowing/lending rate, and P_0 is the current price of the bond. Explain why this formula gives an arbitrage free futures price.

Let f_1 be the futures price and suppose that $f_1 > f$. Then borrow P_0 dollars from the bank, buy the bond and "sell" the contract. At time t receive $f_1 + \frac{RF}{100}t$ dollars in exchange for the bond. Use this to pay of what you owe the bank, which is $P_0\left(1 + \frac{i}{100}t\right)$ dollars. The profit is

$$\begin{aligned} f_1 + \frac{RF}{100}t - P_0\left(1 + \frac{i}{100}t\right) &= f_1 + \frac{P_0 y_c}{100}t - P_0\left(1 + \frac{i}{100}t\right) \\ &= f_1 - P_0\left(1 + \frac{i - y_c}{100}t\right) \\ &= f_1 - f > 0 \end{aligned}$$

If f_1 is less than f reverse the process. "Buy" the contract, short sell the bond for P_0 dollars and invest the money at $i\,\%$. After t years you have $P_0\left(1 + \frac{i}{100}t\right)$ dollars from which you take $f_1 + \frac{RF}{100}t$ dollars in exchange for the bond from the futures contract seller. This bond is then used to close out your short position. The profit from this exchange is

$$\begin{aligned} P_0\left(1 + \frac{i}{100}t\right) - \left(f_1 + \frac{RF}{100}t\right) &= P_0\left(1 + \frac{i}{100}t\right) - \frac{y_c}{100}t - f_1 \\ &= f - f_1 > 0 \end{aligned}$$

7. (10) A Treasury bond which matures in 30 years has a coupon rate of 4.5 percent and a yield of 4.5%. The face value of the bond is 100,000.

   (a) What is the future's price for a futures contract which will mature in 5 months? Assume that the current borrowing/lending rate is 3.5%.

   $$f = 99,583.33$$

   (b) Assume that 2 months from now the bond's yield drops to 4.45% and someone offers you \$50 for the contract. Assuming that the lending/borrowing rate is still 3.5% at that time, is this an arbitrage opportunity? Explanations are definitely required.

   The value of the futures contract in 2 months is $f_\tau = 987.63$ which is considerably more than \$50.

   Let P_τ donote the price of the bond at time τ which is two months from start of futures contract and 3 months till its maturity.

   The arbitrage strategy is as follows. "buy" the contract for \$50, short sell the bond for $P_\tau = 100,815.90$ dollars. Invest $P_\tau - 50$ at i% for 3 months. At the end of the three months pay out $f + \frac{P_0 y_c}{100} t$ dollars for the bond which is then used to close out the short bond position. The profit is

   $$(P_\tau - 50)\left(1 + \frac{i}{100}(t-\tau)\right) - \left(f + \frac{P_0 y_c}{100} t\right) = 945.83.$$

The Maple worksheet which I used follows

```
# Problem 1
# Bond 1
> R1:=5.5;fp1:=101.50;F1:=100;z1:=22;x1:=183;
> ai1:=R1*F1/200*(x1-z1)/x1;
> P1:=fp1+ai1;
> eq1:=(F1+R1*F1/200)=P1*(1+y1/100*(z1/(2*x1)));
> yield1:=solve(eq1,y1);
```

```
                    R1 := 5.5

                   fp1 := 101.50

                    F1 := 100

                    z1 := 22
```

```
                              x1 := 183

                            ai1 := 2.419398907

                            P1 := 103.9193989

          eq1 := 102.7500000 = 103.9193989 + .06246521245 y1

                          yield1 := -18.72080241

# Bond 2
> R2:=5.0;fp2:=103.0;F2:=100;z2:=71;x2:=184;
> ai2:=R2*F2/200*(x2-z2)/x2;
> P2:=fp2+ai2;
> eq2:=P2=(R2*F2/200)*(1+y2/200)^(-z2/x2)+(F2+R2*F2/200)*(1+y2/200)^(-1-
> z2/x2);

                               R2 := 5.0

                              fp2 := 103.0

                               F2 := 100

                                z2 := 71

                               x2 := 184

                            ai2 := 1.535326087

                            P2 := 104.5353261

  eq2 := 104.5353261 =

                                1                 102.5000000
        2.500000000 ------------------- + ------------------
                                /71 \                   255
                                |---|                   ---
                                \184/                   184
                     (1 + 1/200 y2)          (1 + 1/200 y2)

> yield2:=fsolve(eq2,y2);

                          yield2 := .6523412887
```

```
# Problem 2
> R2:=5.0;fp2:=103.0;F2:=100;z2:=71;x2:=184;
> omega1:=R2*F2/200*(1+yield2/200)(-z2/x2)/P2;
> omega2:=(F2+R2*F2/200)*(1+yield2/200)(-1-z2/x2)/P2;
> duration2:=(z2/x2)*omega1+(1+z2/x2)*omega2;

                              R2 := 5.0

                             fp2 := 103.0

                              F2 := 100

                               z2 := 71

                              x2 := 184

                          omega1 := .02399336529

                          omega2 := .9837279773

                         duration2 := 1.372576973

# Problem 3
> newinvoiceprice:=P2*(1-(-0.1)*duration2/(200+yield2));
> newflat:=newinvoiceprice-ai2;

                     newinvoiceprice := 104.6068342

                         newflat := 103.0715081

# Problem 4
> restart:
> R1:=5.5;R2:=5;F2:=100;F1:=100;
> # the following values were obtained using the HP12C
> P1:=100.2439;P2:=100.97078;
> r[1]:=5;
> eq:=P2=(R2*F2)/200*(1+r[1]/200)^(-1)+(F2+R2*F2/200)*(1+r2/200)^(-2);
> r[2]:=fsolve(eq,r2=0..10);

                              R1 := 5.5

                               R2 := 5

                              F2 := 100

                              F1 := 100

                            P1 := 100.2439
```

```
                          P2 := 100.97078

                             r[1] := 5

                                100        205/2
                eq := 100.97078 = --- + ---------------
                                41                   2
                                      (1 + 1/200 r2)


                          r[2] := 3.987623266
```

```
# Problem 5
> restart:
> F:=50000;R:=4;y:=3.75;
> P_0:=500*103.4956;P_1:=500*102.737;# from HP12C
> earned:=P_0*(1+4/200);
> paidOut:=P_1+R*F/200;
> profitlLoss:=earned-paidOut;
```

```
                              F := 50000

                                R := 4

                              y := 3.75

                           P_0 := 51747.8000

                            P_1 := 51368.500

                         earned := 52782.75600

                          paidOut := 52368.500

                        profitlLoss := 414.25600
```

```
# Problem 7b
> P_0:=100000;F:=100000;R:=4.5;y:=4.5;yc:=R*F/100;
> t:=5/12;tau:=2/12;i:=3.5;
> ai:=R*F/100*tau;
> P_tau:=100.8159*1000+ai; #flat price from HP12C
> f_tau:=P_tau-P_0*(1+i*t/100)*(1+i*(t-tau)/100)^(-1);
> profit:=(f_tau-50)*(1+i*(t-tau)/100);
```

```
                              P_0 := 100000

                               F := 100000
```

R := 4.5

y := 4.5

yc := 4500.000000

t := 5/12

tau := 1/6

i := 3.5

ai := 750.0000000

P_tau := 101565.9000

f_tau := 987.6266

profit := 945.8308328