1. (25) The following table describes two coupon bonds, *A* and *B* both of which mature in one year.

Bond	Coupon Rate	Par Value	Present Value
A	8%	1000.00	1,037.60
B	4%	1000.00	998.80

Use this information to determine the discount factors $\delta(1/2)$ and $\delta(1)$

The equations for the present values of the two bonds are:

$$1037.60 = \delta(1)1000 + \sigma(1)40$$
$$998.80 = \delta(1)1000 + \sigma(1)20 \, .$$

The solutions to this system are

$$\sigma(1) = \frac{1037.60 - 998.80}{20} = 1.94$$
$$\delta(1) = \frac{998.80 - 20\sigma(1)}{1000} = 0.96 \, .$$

Thus $\delta(1/2) = 1.94 - .96 = 0.98$.

2. (20) The table below lists discount factors:

t	1/2	1	3/2	2
$\delta(t)$	.99	.98	.96	.92

A forward contract is sold, settlement date in one year. At settlement a zero bond with par value of $1000, and which matures in one more year (that is 2 years from now) will be exchanged for the delivery price of $950. What is the cost of the futures contract, and who should receive the money? The person selling the contract (he turns over the bond) or the person buying the contract (she pays the $950)? Be sure to explain your choice of who receives the money.

A formula relating these quantities is

$$F + \delta(T)D = A,$$

where *F* is the price of the contract, *T* is the time to settlement date, *D* is the delivery price, and *A* is the current value of the asset. Entering the corresponding values into this equation gives us

$$F + \delta(1)950 = \delta(2)1000$$
$$F = 920 - .98 * 950$$
$$= -11.0 \, .$$

That is $\$11.00$ is given to the buyer of the contract. One way to see that the buyer gets the money is to realize that the bond in one year will be worth

$$\frac{\delta(2)1000}{\delta(1)} = \frac{920}{.98} = 938.78$$

and the buyer will have to pay $950 dollars for an asset not even worth $940.

3. (20) An individual invests \$1000 at 8% compounded quarterly for two years, and then invests the total accumulated at 8% compounded monthly.

a. How much will she have after a total of 3 and 1/2 years (2 years at 8% compounded quarterly and 1.5 years compounded monthly)?

$$\text{Total amount after } 3.5 \text{ years} = 1000\left(1+\frac{0.08}{4}\right)^{8}\left(1+\frac{0.08}{12}\right)^{18} = 1320.52$$

b. What constant interest rate, $100r\%$, would give the same result, if interest was compounded continuously?

If interest is compounded continuously at $100r\%$ per year we compute $A(t)$, by the formula $A(t) = A_0e^{rt}$. So we want an r that satisfies

$$1320.52 = 1000e^{3.5r}$$

$$3.5r = \ln\left(\frac{1320.52}{1000}\right) = 0.2780255984$$

$$r = \frac{0.2780255984}{3.5} = 7.943588526 \times 10^{-2}$$

The continuous compounding rate is 7.94%.

4. (20) A bond with a coupon rate of 4% makes coupon payments on February 15$^{\text{th}}$ and August 15$^{\text{th}}$ every year. If you buy the bond on Sept. 25, 2009, what is the amount of accrued interest? Does the buyer or seller of the bond pay the accrued interest? Why?

There are 184 days from August 15 to February 15, and 41 days from August 15 to September 25. The coupon payment is \$20.00. Thus, the accrued interest is

$$\frac{41}{184}20 \approx 4.46\,.$$

The accrued interest is paid to the seller of the bond as he was the owner of the bond for 41 of the days in the coupon period and so is entitled to that portion of the coupon payment that would have been earned during his period of ownership.

5. (10) Show, by an "arbitrage free" argument that the discount factor, $\delta(t)$, must be a strictly decreasing function of t.

Suppose $t_1 < t_2$ and $\delta(t_1) < \delta(t_2)$. Then short sell a par \$1000 zero that matures at t_2 and buy a par \$1000 zero that matures at t_1. You will then have the following amount of money

$$1000(\delta(t_2) - \delta(t_1)) > 0.$$

At t_1 you will receive \$1000 as your long position matures, and at time t_2 use this \$1000 to close out your short position. Moreover, since you have this \$1000 available to invest from t_1 until t_2, you can invest this and generate some income. Thus, even if $\delta(t_1) = \delta(t_2)$, you still can generate free money. Thus, not only is $\delta(t_1) < \delta(t_2)$ not possible, they also cannot equal each other.

6. (5) Show that the function $f(x) = \dfrac{\ln(1+x)}{x}$ is a strictly decreasing function of x for $x > 0$.

Differentiating f we get

$$
\begin{aligned}
f'(x) &= \frac{1}{x((x+1))} - \frac{\ln(1+x)}{x^2} \\
&= \frac{1}{x^2(x+1)}[x - (1+x)\ln(x+1)] \;.
\end{aligned}
$$

Since $x > 0$, the sign of f' is exactly the same as the sign of the expresion $[x - (1+x)\ln(x+1)]$. Call this expression $g(x)$. Note that the limit of g as x approaches zero from above is 0, and that the derivative of g equals

$$g'(x) = -\ln(x+1) \;.$$

Since $x > 0$, $\ln(x+1) > 0$, Thus, g' is negative, which means that g is a decreasing function, and since the limit from above at $x = 0$ of g is zero, this forces $g < 0$. Thus f' is negative too, which means that f is a decreasing funcion of x.