Explanations are required in order to receive any credit. No explanation, zero points.

1. (60) Suppose the six month forward rates are: $f(1/2) = 0.04$, $f(1) = 0.04$, $f(3/2) = 0.04$.

a. Determine the current price of a coupon bond, par value $1000, coupon rate 6%, with 18 months to maturity.

We need the discount factors to determine the present value of the bond. The six month forward rates are related to the discount factors by the equation

$$1 + \frac{f(t)}{2} = \frac{\delta(t-1/2)}{\delta(t)}.$$

Thus,

$$\delta(1/2) = \frac{1}{1+f(1/2)/2} = \frac{1}{1.02} = 0.98039$$
$$\delta(1) = \frac{\delta(1/2)}{1+f(1)/2} = \frac{1}{(1.02)^2} = 0.96116$$
$$\delta(3/2) = \frac{\delta(1)}{1+f(3/2)/2} = \frac{1}{(1.02)^3} = 0.94232$$

Thus, the present value of the bond is

$$\begin{aligned} V &= \delta(3/2)1000 + \sigma(3/2)30 \\ &= 942.32 + (2.88387)30 \\ &= 1028.80 \end{aligned}$$

b. Assuming that you can reinvest at the current 6 month forward rates, what is the total return of the bond?

The total return equals

$$\text{Total return} = \frac{1028.80}{\delta(3/2)} = \frac{1028.80}{0.94232} \approx 1091.77$$

c. What formula describes the yield to maturity for the bond in part a?

$$\begin{aligned} 1028.80 &= 1000\left(1+\frac{y}{2}\right)^{-3} + 30\left(\left(1+\frac{y}{2}\right)^{-1} + \left(1+\frac{y}{2}\right)^{-2} + \left(1+\frac{y}{2}\right)^{-3}\right) \\ &= 1000\left(1+\frac{y}{2}\right)^{-3} + \frac{60}{y}\left(1-\left(1+\frac{y}{2}\right)^{-3}\right) \end{aligned}$$

d. What is the yield for the bond in part a? Hint: you cannot solve the equation.

To solve this problem observe that the forward rates are constant, which implies that

$$\delta(i/2) = \left(1+\frac{0.04}{2}\right)^{-i},$$

and this tells us that the yield to maturity must equal the constant 6 month forward rate, which is 4%.

2. (20) You have purchased a floating rate note with par value \$2000, which matures in 2 years. Assume that

$$\delta(\tau, t) = \frac{1 - 0.04t}{1 - 0.04\tau} \text{ for } 0 \le \tau \le t \le 5 .$$

a. What payments will you receive from this note over the next two years?

Let $r(t)$ donote the six month spot rate in effect at time $t - 1/2$. Then we have the following formula

$$1 + \frac{r(t)}{2} = \frac{1}{\delta(t - 1/2, t)} = \frac{1.02 - 0.04t}{1 - 0.04t}$$

$$r(t) = 2\left(\frac{1.02 - 0.04t}{1 - 0.04t} - 1\right) = \frac{0.04}{1 - 0.04t} .$$

Thus,

$$r(1/2) = \frac{0.04}{1 - 0.02} = 0.04081$$

$$r(1) = \frac{0.04}{1 - 0.04} = 0.04166$$

$$r(3/2) = \frac{0.04}{1 - 0.06} = 0.04255$$

$$r(2) = \frac{0.04}{1 - 0.08} = 0.04347$$

The table below shows the payments

t	1/2	1	3/2	2
$r(t)$	0.04081	0.04166	0.04255	0.04347
Payment	40.81	41.66	42.55	43.47 + 2000

b. What would be the price of this note if you purchase it three months from now? That is, you purchase the note when there are 21 months to maturity.

The price of this note equals the value of a zero at $t = 1/4$ with par value 2,040.81, which matures at $t = 1/2$. The value of this zero is

$$Z = 2,040.81 * \delta(1/4, 1/2)$$

$$= 2,040.81 * \frac{1 - 0.02}{1 - 0.01}$$

$$= 2020.20$$

3. (20) If someone borrows $\$V$ at $100a\%$ interest compounded monthly and agrees to make payments of $\$L$ per month, then the balance after i months, or i payments is

$$B(i) = V\left(1+\frac{a}{12}\right)^i - L\left(\frac{12}{a}\right)\left(\left(1+\frac{a}{12}\right)^i - 1\right).$$

a. Derive this formula.

If we set $i = 0$, in this formula we get $B(0) = V$, which is certainly correct. Setting $i = 1$, we have $B(1) = V\left(1+\frac{a}{12}\right)^1 - L\left(\frac{12}{a}\right)\left(\left(1+\frac{a}{12}\right)^1 - 1\right)$

$$\begin{aligned} B(1) &= V\left(1+\frac{a}{12}\right)^1 - L\left(\frac{12}{a}\right)\left(\left(1+\frac{a}{12}\right)^1 - 1\right) \\ &= V\left(1+\frac{a}{12}\right) - L, \end{aligned}$$

and this is exactly the balance owed after 1 payment. Next is the induction step. That is, assume the result is true for $i = k$, and show it is true for $i = k+1$.

$$\begin{aligned} \text{Balance after } k+1 \text{ payments } &= B(k)\left(1+\frac{a}{12}\right) - L \\ &= \left[V\left(1+\frac{a}{12}\right)^k - L\left(\frac{12}{a}\right)\left(\left(1+\frac{a}{12}\right)^k - 1\right)\right]\left(1+\frac{a}{12}\right) - L \\ &= V\left(1+\frac{a}{12}\right)^{k+1} - L\left(\frac{12}{a}\right)\left(1+\frac{a}{12}\right)^{k+1} + L\left(\frac{12}{a}\right)\left(1+\frac{a}{12}\right) - L \\ &= V\left(1+\frac{a}{12}\right)^{k+1} - L\left(\frac{12}{a}\right)\left(1+\frac{a}{12}\right)^{k+1} + L\left(\frac{12}{a}\right) \\ &= V\left(1+\frac{a}{12}\right)^{k+1} - L\left(\frac{12}{a}\right)\left(\left(1+\frac{a}{12}\right)^{k+1} - 1\right) \\ &= B(k+1). \end{aligned}$$

b. How large must L be in order for the balance to decrease with each payment. That is, what condition on L guaranties that $B(i) > B(i+1)$.

The easiest way is to realize that the first payment must decrease the balance. That is,

$$\begin{aligned} V\left(1+\frac{a}{12}\right) - L &< V \\ \frac{a}{12}V &< L. \end{aligned}$$

If the balance decreases after the first payment, then, since L is constant, all balances will decrease. Thus, we need $L > \frac{a}{12}V$, which is the statement that L must be greater than the interest the debt incurs.