1. (15) The discount factors $\delta(t)$ tell you how much a dollar at time $t \geq 0$ is worth today. Let $\delta(\tau, t)$ be what a dollar at time t is worth at time τ, $0 \leq \tau \leq t$.

a. Assume that interest rates do not change find a formula expressing $\delta(\tau, t)$ in terms of the discount factors $\delta(t)$.

Since $\delta(t)$ is the present value of 1 dollar at time t, $\frac{1}{\delta(t)}$ is what a dollar at time 0 is worth at time t. Thus, the expression $\frac{\delta(t)}{\delta(\tau)}$ must give us the value at time τ of a dollar at time t. That is,

$$\delta(\tau, t) = \frac{\delta(t)}{\delta(\tau)}$$

b. Let $f(h, t)$ denote the h year rate paid $t - h$ years forward. Assuming that interest rates are constant express $f(h, t)$ in terms of $\delta(\tau, t)$.

The formula relating the forward rates to the discount factors is

$$\left(1 + \frac{f(h, t)}{2}\right)^{2h} = \frac{\delta(t-h)}{\delta(t)} = \frac{1}{\delta(t-h, t)}.$$

c. If $r(t)$ denotes the 6 month spot rate in effect $t - 1/2$ years forward, express $r(t)$ in terms of $\delta(\tau, t)$.

If interest rates are constant the the 6 month forward rate t years forward will equal the spot rate $r(t)$. Thus,

$$1 + \frac{r(t)}{2} = \frac{1}{\delta(t - 1/2, t)}$$

2. (20) A bond with par value \$1000 and coupon rate 4% matures in 5 years. A forward contract for this bond with settlement date in 3 years is sold for \$100. What is the delivery price? Assume that $\delta(t) = 1 - 0.04t$, for $0 \leq t \leq 5$.

The first thing to do is to compute the present value of the bond, but only using bond payments that will be made after the delivery date. Let A donote this value, then

$$\begin{aligned} A &= \delta(3.5)20 + \delta(4)20 + \delta(4.5)20 + \delta(5)1020 \\ &= 866.40\ . \end{aligned}$$

A formula relating the delivery price, A, and the price of the contract is $F + \delta(T)D = A$. Thus,

$$\begin{aligned} D &= \frac{A - F}{\delta(T)} \\ &= \frac{866.40 - 100}{\delta(3)} \\ &= 870.91 \end{aligned}$$

3. (15) Assume $\delta(t) = 1 - 0.04t$, and that $r(t)$, which represents the six month spot rate in effect $t - 1/2$ years in the future equals $r(t) = \frac{0.04}{1-0.04t}$. Both of these equations are valid for $0 \leq t \leq 10$. Let A denote a coupon bond with par value $1000, a coupon rate of 4%, and 5 years to maturity.

a. Calculate the present value of bond A.

The present value of A equals

$$
\begin{aligned}
V = {} & 20(\delta(1/2) + \delta(1) + \delta(1.5) + \delta(2) + \delta(2.5) + \delta(3) + \delta(3.5) + \delta(4) + \delta(4.5) + \delta(5)) \\
& + \delta(5)1000 \\
= {} & 20(8.9) + 0.8 \times 1000 \\
= {} & 978.00
\end{aligned}
$$

b. Assuming you invest all of the cash flows from bond A in a rollover account at the 6 month spot rates in effect at the time you receive the cash. How much money will you have after 5 years?

The total return is computed by calculating how much money you'll have from investing $20 for 4.5 years, 4 years, 3.5 years, etc., and then adding these amounts up. Note, a formula for this is

$$A(k) = 20 \prod_{i=k+1}^{10} \left(1 + \frac{r(i/2)}{2}\right) \text{ for } k = 1, \cdots, 9,$$

where $A(k)$ denotes the total return of a $20 coupon payment received at $t = k/2$. Using this formula we have

$$A(1) = 24.50, A(2) = 24.00, \cdots, A(9) = 20.50$$

The total return is the sum of the preceeding values plus the coupon payment and par payment at $t = 5$. The total return equals

$$
\begin{aligned}
\text{total return} &= \sum_{i=1}^{9} A(i) + 1020.00 \\
&= 202.50 + 1020.00 \\
&= 1222.50
\end{aligned}
$$