1. (15) Define the following:

   (a) A boundary point of a set S.

   The point z_0 is a boundary point of a set S if every epsilon neighborhood of z_0 contains a point of S and a point not in S.

   (b) The imaginary part of a complex number.

   If $z = (x, y) = x + iy$, where x and y are real numbers, then the imaginary part of z is the real number y.

   (c) $Arg\,(z)$, where z is a complex number.

   $Arg\,(z)$ is that number θ such that $-\pi < \theta \le \pi$, and

$$z = |z|\,e^{i\theta} = |z|\,(\cos\theta + i\sin\theta)\ .$$

   (d) $\dfrac{df}{dz}$

$$\frac{df}{dz} = \lim_{h\to 0}\frac{f\,(z+h) - f\,(z)}{h}$$

2. (15) Compute the following:

   (a) If $z = \dfrac{-1 + i\sqrt{3}}{2} = re^{iArg(z)}$, find r and $Arg\,(z)$.

$$\begin{aligned} r &= \left(\left(\frac{-1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2\right)^{1/2} = 1 \\ Arg\,(z) &= \frac{\pi}{2} + \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3} \end{aligned}$$

   (b) $1^{1/5}$ Remember this notation means the set of z such that $z^5 = 1$.

$$\begin{aligned} 1 &= e^{2k\pi i} \text{ for } k = 0,\ \pm 1,\ \pm 2,\ \cdots \\ 1^{1/5} &= e^{2k\pi i/5} \text{ for } k = 0,\ \pm 1,\ \pm 2,\ \cdots \\ &= \left\{e^{2\pi i/5},\ e^{4\pi i/5},\ e^{6\pi i/5},\ e^{8\pi i/5},\ 1\right\} \end{aligned}$$

3. (20) The function $\sqrt{z} = r^{1/2}e^{iArg(z)/2}$, where $r = |z|$, is called the square root of z.

   (a) Show that $(\sqrt{z})^2 = z$ for any complex number z.

$$\begin{aligned} \left(\sqrt{z}\right)^2 &= \left(r^{1/2}e^{iArg(z)/2}\right)^2 \\ &= \left(r^{1/2}\right)^2\left(e^{iArg(z)/2}\right)^2 \\ &= (r)\left(e^{iArg(z)}\right) \\ &= z \end{aligned}$$

(b) Where is $\sqrt{z}$ a discontinuous function of z? Be sure to explain why.

The function $\sqrt{z}$ is discontinuous on the negative real axis. To see that this function is discontinuous at any point of the form $z = x$, where x is negative. Note that $Arg\,(x) = \pi$, but that $Arg(x + iy)$ is negative when $y < 0$, and approaches $-\pi$ as y goes to zero through negative values. Thus,

$$\begin{aligned}
\lim_{y \to 0^-} \sqrt{x+iy} &= \lim_{y \to 0^-} \left(\left(x^2+y^2\right)^{1/4} \left(\cos \frac{(Arg(x+iy)}{2} + i \frac{\sin\left(Arg(x+iy)\right)}{2} \right) \right) \\
&= \left|x^2\right|^{1/4} \left(\cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right) \right) \\
&= |x|^{1/2}\,(-i)
\end{aligned}$$

However,

$$\begin{aligned}
\sqrt{x} &= |x|^{1/2} \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right) \\
&= |x|^{1/2}\, i
\end{aligned}$$

Since the limiting value of $\sqrt{z}$ is not the same as the function value, as long as $x < 0$, we see that $\sqrt{z}$ is discontinuous at every point on the negative real axis.

4. (20) If $f\,(z) = u + iv$ is an analytic function, with u and v the real and imaginary parts of f respectively, the functions u and v satisfy a set of equations called the Cauchy Riemann equations. Derive these equations (Cartesian coordinates).

The Cauchy Riemann equations follow from the fact that if $f'\,(z)$ exists, then its value must be the same regardless of how Δz appoaches zero. The lines below calculate the derivative of f by first letting Δz appoaches zero through real values and then through imaginary values.

Set $f = u + iv$, and $\Delta z = h$, with h a real number, then

$$\begin{aligned}
\frac{df}{dz} &= \lim_{h \to 0} \frac{u\,(x+h,y) + iv\,(x+h,y) - u\,(x,y) - iv\,(x,y)}{h} \\
&= \lim_{h \to 0} \frac{u\,(x+h,y) - u\,(x,y)}{h} + \lim_{h \to 0} \frac{iv\,(x+h,y) - iv\,(x,y)}{h} \\
&= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}
\end{aligned}$$

Now set $\Delta z = ih$, with h a real number, then

$$\begin{aligned}
\frac{df}{dz} &= \lim_{h \to 0} \frac{u\,(x,y+h) + iv\,(x,y+h) - u\,(x,y) - iv\,(x,y)}{ih} \\
&= \lim_{h \to 0} \frac{u\,(x,y+h) - u\,(x,y)}{ih} + \lim_{h \to 0} \frac{iv\,(x,y+h) - iv\,(x,y)}{ih} \\
&= \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}
\end{aligned}$$

Two complex numbers are equal if and only if they have the same real and imaginary parts. Thus,

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}
\end{aligned}$$

5. (10) Let $f(z) = \dfrac{z-1}{z+1}$. Let D be the set of z such that $Re(z) \geq 0$. To what set does f map the set D? Hint: what does f do to the y-axis?

There are two ways to see what f does to the region D. The first and easiest is to observe that the absolute value of the numerator measures the distance from z to 1, and the absolute value of the denominator measures the distance from z to -1. The imaginary axis is the locus of all points equidistant from 1 and -1. Thus $|f(iy)| = 1$, and if z lies in the right half plane then z is clearly closer to 1 than it is to -1. Thus, if $Re(z) > 0$, then $|f(z)| < 1$. So f maps the set D into the set $|z| \leq 1$.

A second method is to compute $f(iy)$

$$\begin{aligned} f(iy) &= \frac{iy-1}{iy+1} = \frac{(iy-1)(-iy+1)}{1+y^2} \\ &= \frac{-1+y^2+2iy}{1+y^2} \end{aligned}$$

Setting $u = \dfrac{-1+y^2}{1+y^2}$ and $v = \dfrac{2y}{1+y^2}$ we see that

$$\begin{aligned} u^2+v^2 &= \left(\frac{-1+y^2}{1+y^2}\right)^2 + \left(\frac{2y}{1+y^2}\right)^2 \\ &= \frac{y^4-2y^2+1+4y^2}{(1+y^2)^2} = \frac{y^4+2y^2+1}{(1+y^2)^2} \\ &= \frac{(y^2+1)^2}{(y^2+1)^2} = 1. \end{aligned}$$

So this map sends the imaginary axis to the unit circle. We also note that it maps 1 to the origin. So the right half plane is mapped onto the interior of the unit circle. In fact the image of any line of the form $a+iy$, for any fixed a has the form

$$\begin{aligned} f(a+iy) &= \frac{(a-1)+iy}{(a+1)+iy} \\ &= \left[\frac{(a-1)+iy}{(a+1)+iy}\right]\left[\frac{(a+1)-iy}{(a+1)-iy}\right] \\ &= \frac{(a^2-1)+y^2+2iy}{(a+1)^2+y^2}, \end{aligned}$$

and it (tediously) can be shown that this is a circle centered at the point $z_0 = \dfrac{a}{a+1}$ with radius $\dfrac{1}{a+1}$. Geometrically this means that every vertical line in the right half plane gets mapped to a circle which is inside the unit circle, is centered on the real axis and passes through the point $z = 1$.

6. (20) Let $u(x,y) = 2x - x^3 + 3xy^2$.

(a) Show that u is an harmonic function.

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(2 - 3x^2 + 3y^2\right) = -6x \\ \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}(6xy) = 6x\end{aligned}$$

Thus, we have $\nabla^2 u = 0$.

(b) Find an harmonic conjugate, v, of u.

$$\begin{aligned}\text{From } \frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2 \\ \text{we have } v &= 2y - 3x^2 y + y^3 + \phi(x)\end{aligned}$$

and from

$$\begin{aligned}\frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} = -(6xy) \\ \text{we have } -6xy + \phi'(x) &= -6xy\end{aligned}$$

Thus, $\phi' = 0$ or ϕ is a constant function. So

$$v = 2y - 3x^2 y + y^3$$

is one of the many harmonic conjugates of u.

(c) Explain why the function $f(z) = u + iv$, where u and v are as in parts (a) and (b), is an analytic function.

Since the functions u and v satisfy the Cauchy Riemann equations and are continuously differentiable, we know that $f = u + iv$ is analytic. Incidentally you can show that $f = 2z - z^3$