1. (12) Define the following:

(a) $\int_\Gamma f(z)\, dz$, where Γ is a smooth curve with finite length.

Let $f = u + iv$, and let Γ be parmetrized by $\Gamma = (x(t), y(t)) = x(t) + iy(t)$ for $a \le t \le b$. Then

$$\int_\Gamma f(z)\, dz = \int_a^b \left(u\frac{dx}{dt} - v\frac{dy}{dt} \right) dt + i \int_a^b \left(u\frac{dy}{dt} + v\frac{dx}{dt} \right) dt$$

The parametrization of Γ is assumed to be smooth and the functions u and v are assumed to be continuous functions of x and y.

(b) The function f is analytic at z_0.

f is analytic at z_0 if there is a neighborhood of z_0 such that f is differentiable at every point in this neighborhood.

(c) The function f has a pole of order 4 at z_0.

The statement means that z_0 is an isolated singular point of f, and that the Laurent expansion of f about the point z_0 has the form

$$\begin{aligned} f(z) &= \frac{b_4}{(z-z_0)^4} + \frac{b_3}{(z-z_0)^3} \frac{b_2}{(z-z_0)^2} + \frac{b_1}{(z-z_0)^1} \\ &\quad + \sum_{n=0}^{\infty} a_n (z-z_0)^n \end{aligned}$$

with the coefficienct $b_4 \ne 0$.

(d) The residue of f at z_0.

The residue of f at z_0 is the coefficient of $(z-z_0)^{-1}$ in the Laurent expansion of f.

2. (8) If $f(z) = u + iv$ has a derivative at $z_0 = x_0 + iy_0$, then the real valued functions u and v satisfy a pair of partial differential equations at z_0. These equations are called the Cauchy-Riemann equations.

(a) State the Cauchy-Riemann equations.

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}$$

(b) Prove that u and v must satisfy these equations.

The Cauchy Riemann equations follow from the fact that if $f'(z)$ exists, then its value must be the same regardless of how Δz appoaches zero. The lines below calculate the derivative of f by first letting Δz appoaches zero through real values and then through imaginary values.

Set $f = u + iv$, and $\Delta z = h$, with h a real number, then

$$\begin{aligned}
\frac{df}{dz} &= \lim_{h\to 0} \frac{u(x+h,y) + iv(x+h,y) - u(x,y) - iv(x,y)}{h} \\
&= \lim_{h\to 0} \frac{u(x+h,y) - u(x,y)}{h} + \lim_{h\to 0} \frac{iv(x+h,y) - iv(x,y)}{h} \\
&= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}
\end{aligned}$$

Now set $\Delta z = ih$, with h a real number, then

$$\begin{aligned}
\frac{df}{dz} &= \lim_{h\to 0} \frac{u(x,y+h) + iv(x,y+h) - u(x,y) - iv(x,y)}{ih} \\
&= \lim_{h\to 0} \frac{u(x,y+h) - u(x,y)}{ih} + \lim_{h\to 0} \frac{iv(x,y+h) - iv(x,y)}{ih} \\
&= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}
\end{aligned}$$

Two complex numbers are equal if and only if they have the same real and imaginary parts. Thus,

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}
\end{aligned}$$

3. (15)

(a) State the Cauchy-Goursat theorem. Be sure to include any needed hypotheses.

Suppose $f(z)$ is analytic inside and on a simple closed contour Γ. Then

$$\int_{\Gamma} f(z)\, dz = 0$$

(b) Suppose f has singular points z_1 and z_2 inside a closed contour C. Why does $\int_C f(z)\, dz = 2\pi i \left(\text{Res}_{z=z_1} f + \text{Res}_{z=z_2} f\right)$

Let C_1 and C_2 denote positively oriented circles centered at z_1 and z_2 respectively, such that both circles and their interiors are disjoint from each other, and both are completely contained inside the original contour C. Then the Cauchy-Goursat theorem is used to show that

$$\int_C f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz$$

The value of the contour integral of f over these positively oriented circles is then computed by looking at the Laurent series expansion of f around each of these points. We parametrize the circle C_i by $z(t) = z_i + \rho e^{i\theta}$, $0 \leq \theta \leq 2\pi$. Then we have

$$\begin{aligned}
\int_{C_i} f(z)\, dz &= \int_{C_i} \left(\sum_{n=-\infty}^{\infty} c_n (z - z_i)^n \right) dz \\
&= \sum_{n=-\infty}^{\infty} c_n \int_{C_i} (z - z_i)^n\, dz \\
&= \sum_{n=-\infty}^{\infty} c_n i \int_0^{2\pi} \rho^{n+1} e^{i(n+1)\theta}\, d\theta
\end{aligned}$$

The only one of the integrals that is not equal to zero is the one in which the exponent in $e^{i(n+1)\theta}$ is zero. That is, $n + 1 = 0$ or $n = -1$. Thus,

$$\begin{aligned}
\int_{C_i} f(z)\, dz &= (c_{-1})\, i \int_0^{2\pi} d\theta \\
&= 2\pi i\, (c_{-1}) \\
&= 2\pi i \text{Res}_{z=z_i} f
\end{aligned}$$

4. (20) Let $f(z) = \dfrac{\sin z}{z(z-1)(z-2)^2}$.

(a) Locate all of the isolated singular points of f and determine their type.

The isolated singular points of f are $z = 0$, 1, and 2. Since $\sin z$ is zero at $z = 0$, and not equal to zero at the other two singular points, 0 is a removable singularity, $z = 1$ is a simple pole, that is a pole of order 1, and $z = 2$ is a pole of order 2.

(b) Calculate the residue of f at each of its isolated singular points.

At $z = 0$ the residue of f is zero.

At $z = 1$ the residue equals

$$\begin{aligned} \mathrm{Res}_{z=1} f &= \lim_{z \to 1} (z-1) \frac{\sin z}{z(z-1)(z-2)^2} \\ &= \lim_{z \to 1} \frac{\sin z}{z(z-2)^2} = \frac{\sin 1}{1} \approx 0.84 \end{aligned}$$

At $z = 2$ the residue equals

$$\begin{aligned} \mathrm{Res}_{z=2} f &= \lim_{z \to 2} \frac{d}{dz} \left[(z-2)^2 \frac{\sin z}{z(z-1)(z-2)^2} \right] \\ &= \lim_{z \to 2} \frac{d}{dz} \left[\frac{\sin z}{z(z-1)} \right] \\ &= \lim_{z \to 2} \frac{(\cos z) z (z-1) - \sin z (2z-1)}{z^2 (z-1)^2} \\ &= \frac{2\cos 2 - 3\sin 2}{4} \approx -0.89 \end{aligned}$$

5. (20) Let $f(z) = z^2\left(e^{1/z} - 1\right)$.

(a) Find the Laurent series expansion of f about $z = 0$. What type of singular point is 0 ?

$$\begin{aligned} f(z) &= z^2\left(\sum_{n=0}^{\infty} \frac{1}{z^n n!} - 1\right) \\ &= \sum_{n=1}^{\infty} \frac{1}{z^{n-2} n!} \\ &= z + \frac{1}{2} + \frac{1}{z(3!)} + \frac{1}{z^2(4!)} + \cdots + \frac{1}{z^n(n+2)!} + \cdots \end{aligned}$$

Since the number of terms in which z appears to a negative exponent is not finite, 0 is an essential singular point for f.

(b) Let C_2 denote the positively oriented circle of radius 2 centered at the origin. Calculate $\int_{C_2} f(z)\, dz$.

$$\begin{aligned} \int_{C_2} f(z)\, dz &= 2\pi i\left(\operatorname{Res}_{z=0} f\right) \\ &= 2\pi i\left(\frac{1}{6}\right) = \frac{\pi i}{3} \end{aligned}$$

6. (15) Let $f(z) = \dfrac{z^2}{1+z^4}$ and let C_R denote the upper half of the positively oriented circle of radius R centered at the origin.

(a) Show $\lim\limits_{R\to\infty} \int_{C_R} f(z)\, dz = 0$.

If $|z| = R$ then $\left|\dfrac{z^2}{1+z^4}\right| = \dfrac{R^2}{|1+z^4|} \le \dfrac{R^2}{R^4-1}$. Thus,

$$\left|\int_{C_R} f(z)\, dz\right| \le \frac{R^2}{R^4-1}(\pi R) = \frac{\pi R^3}{R^4-1}$$

Thus, as R gets arbitrarily large the absolute value of the integral approaches zero. That is, $\lim\limits_{R\to\infty} \int_{C_R} f(z)\, dz = 0$.

(b) Using residue theory calculate $\int_0^\infty \dfrac{x^2}{1+x^4}dx$. Hint: the integrand is an even function of x.

Let Γ_R denote the closed contour that consists of that part of the real line between $-R$ and R, and the half circle C_R. Then we have

$$\begin{aligned}
\int_0^\infty \frac{x^2}{1+x^4}dx &= \frac{1}{2}\int_{-\infty}^\infty \frac{x^2}{1+x^4}dx \\
&= \frac{1}{2}\lim_{R\to\infty}\left(\int_{-R}^{R} \frac{x^2}{1+x^4}dx + \int_{C_R}\frac{z^2}{1+z^4}\,dz\right) \\
&= \frac{1}{2}\lim_{R\to\infty}\left(\int_{\Gamma_R}\frac{z^2}{1+z^4}\,dz\right)
\end{aligned}$$

For $R > 1$ the only singular points of the integrand are at $z_1 = e^{\pi i/4}$ and $z_2 = e^{3\pi i}/4$. These are simple poles of the integrand. Thus,

$$\begin{aligned}
\int_0^\infty \frac{x^2}{1+x^4}dx &= \frac{1}{2}2\pi i\left(\mathrm{Res}_{z=z_1}\left[\frac{z^2}{1+z^4}\right] + \mathrm{Res}_{z=z_2}\left[\frac{z^2}{1+z^4}\right]\right) \\
&= \pi i\left(\frac{z_1^2}{4z_1^3}+\frac{z_2^2}{4z_2^3}\right) = \frac{\pi i}{4}\left(\frac{1}{z_1}+\frac{1}{z_2}\right) \\
&= \frac{\pi i}{4}\left(e^{-\pi i/4}+e^{-3\pi i}\right) = \frac{\pi i}{4}\left(-2i\sin\left(\frac{\pi}{4}\right)\right) \\
&= \frac{\sqrt{2}\pi}{4}
\end{aligned}$$

7. (10) Let C_ρ denote the top half of the circle of radius ρ centered at the origin. Suppose that C_ρ is traced out from right to left. Calculate $\displaystyle\lim_{\rho\to 0^+}\int_{C_\rho}\frac{\sin z}{z^2}\,dz$.

The upper half circle is parametrized by $z = \rho e^{i\theta}$, $0 \le \theta \le \pi$. The Laurent series expansion of $\sin z/z^2$ about $z = 0$ is

$$\begin{aligned}
\frac{\sin z}{z^2} &= \frac{1}{z^2}\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n-1}}{(2n+1)!}\\
&= \frac{1}{z}+\sum_{n=1}^{\infty}(-1)^n\frac{z^{2n-1}}{(2n+1)!}
\end{aligned}$$

Thus, the contour integral equals

$$\begin{aligned}
\int_{C_\rho}\frac{\sin z}{z^2}\,dz &= \int_0^{\pi}\left[\frac{1}{\rho e^{i\theta}}+\sum_{n=1}^{\infty}(-1)^n\frac{\left(\rho e^{i\theta}\right)^{2n-1}}{(2n+1)!}\right]i\rho e^{i\theta}\,d\theta\\
&= \int_0^{\pi} i\,d\theta + i\sum_{n=1}^{\infty}\frac{(-1)^n\rho^{2n}}{(2n+1)!}\int_0^{\pi}e^{2ni\theta}\,d\theta\\
&= \pi i
\end{aligned}$$

Note: for this particular function there is no need to actually take the limit as ρ tends to zero from above as each of the integrals $\int_0^{\pi}e^{2ni\theta}\,d\theta = 0$. Thus,

$$\lim_{\rho\to 0^+}\int_{C_\rho}\frac{\sin z}{z^2}\,dz. = \pi i$$