Math 407 Solutions Exam 2 March 28, 2011

1. (30) Let Γ denote a curve in $\mathbb{C}$, and let $\gamma(t) = (x(t), y(t))$ for $a \leq t \leq b$, be a parametric representation for Γ. Suppose that each of the coordinate functions of γ is smooth. That is, both x and y have continuous derivatives on the interval $[a, b]$.
Suppose $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$ is a continuous function of z for $z \in \Gamma$.

   (a) Define, using Riemann sums, $\displaystyle\int_\Gamma f(z)\,dz$.

   Let $P = \{t_k\}_{k=0}^{n}$ be a partition of the interval $[a, b]$. A Riemann sum for this integral is any sum of the form

$$\sum_{k=1}^{n} f(z_{\tau_k})(z_k - z_{k-1}) = \sum_{k=1}^{n} f(\gamma(\tau_k))(\gamma(t_k) - \gamma(t_{k-1}))$$

where τ_k lies between t_{k-1} and t_k and the point z_{τ_k}

lies on the curve between the two points $\gamma(t_{k-1}) = z_{k-1}$

and $\gamma(t_k) = z_k$.

$$\begin{aligned}
&= \sum_{k=1}^{n} \left(u(x(\tau_k), y(\tau_k)) + iv(x(\tau_k), y(\tau_k))\right) \left([x(t_k) - x(t_{k-1})] + i[y(t_k) - y(t_{k-1})]\right) \\
&= \sum_{k=1}^{n} u(x(\tau_k), y(\tau_k))[x(t_k) - x(t_{k-1})] - v(x(\tau_k), y(\tau_k)[y(t_k) - y(t_{k-1})] \\
&\qquad + i\sum_{k=1}^{n} u(x(\tau_k), y(\tau_k))[y(t_k) - y(t_{k-1})] + v(x(\tau_k), y(\tau_k)[x(t_k) - x(t_{k-1})] \\
&= \sum_{k=1}^{n} \left(u(x(\tau_k), y(\tau_k))x'(\tau_k) - v(x(\tau_k), y(\tau_k)y'(\tau_k)\right)(t_k - t_{k-1}) \\
&\qquad + i\sum_{k=1}^{n} \left(u(x(\tau_k), y(\tau_k))y'(t_k) + v(x(\tau_k), y(\tau_k)x'(\tau_k)\right)(t_k - t_{k-1}) + \text{ error}\,.
\end{aligned}$$

It is not to hard to show that the 'error' term goes to zero as $n \to \infty$. Thus, we have

$$\begin{aligned}
\int_\Gamma f(z)\,dx &= \lim_{n\to\infty} \sum_{k=1}^{n} f(z_{\tau_k})(z_k - z_{k-1}) \\
&= \int_\Gamma u\,dx - v\,dy + i\int_\Gamma v\,dx + u\,dy\,.
\end{aligned}$$

(b) Let $f(z) = \dfrac{1}{z-i}$, and Γ represent the straight line joining the point 0 to 1. Calculate the integral $\displaystyle\int_\Gamma f(z)\,dz$. You cannot do this by finding an antiderivative of f and then using it to evaluate the integral. You must do this by parametrizing the path and calculating the resulting integral.

$$\begin{aligned}
\int_\Gamma \frac{dz}{z-i} &= \int_0^1 \frac{dx}{x-i} = \int_0^1 \frac{x+i}{1+x^2}\,dx = \int_0^1 \frac{x}{1+x^2}\,dx + i\int_0^1 \frac{1}{1+x^2}\,dx \\
&= \frac{1}{2}\ln(1+x^2)\Big|_0^1 + i\tan^{-1}x\Big|_0^1 = \ln\sqrt{2} + i\tan^{-1}1 \\
&= \ln\sqrt{2} + i\frac{\pi}{4}\,.
\end{aligned}$$

(c) Show that $\left|\displaystyle\int_\Gamma f(z)\,dz\right| \le \displaystyle\int_\Gamma |f(z)|\,|dz|$. Be sure to explain what $|dz|$ means.

$$\begin{aligned}
\left|\int_\Gamma f(z)\,dz\right| &= \lim_{n\to\infty}\left|\sum_{k=1}^n f(z_{\tau_k})(z_k - z_{k-1})\right| \le \lim_{n\to\infty}\sum_{k=1}^n |f(z_{\tau_k})|\,|\Delta z_k| \\
&= \lim_{n\to\infty}\sum_{k=1}^n |f(z_{\tau_k})|\,\big((\Delta x_k|)^2 + (\Delta y_k)^2\big)^{1/2} \\
&= \lim_{n\to\infty}\sum_{k=1}^n |f(z_{\tau_k})|\,\big((x')^2 + (y')^2\big)^{1/2}\,\Delta t = \int_\Gamma |f(z)|\,|dz|
\end{aligned}$$

The term $|dz|$ represents the expression $\big((x')^2 + (y')^2\big)^{1/2}\,dt$, which is commonly denoted as ds, and from real variable calculus if Γ is a curve, which has length L, we know that

$$L = \int_\Gamma ds\,.$$

2. (30) Let $f(z) = u + iv$ be analytic on the closed unit disk. That is, there is a domain D containing the set $S = \{z : |z| \le 1\}$, on which f is analytic. Suppose that
$\sup\{|f(z)| : |z| = 1\} = 1$.

   (a) What can you say about $|f(z)|$ for $z \in S$?

   The maximum modulus principle says that $|f(z)|$ must also be less than 1 for every $z \in S$.

(b) Show that $|f^{(n)}(z)| \leq \dfrac{n!}{(1-|z|)^{n+1}}$ for $z \in S$, where $f^{(n)}(z)$ denotes the nth derivative of f.

Since f is analytic on the set S, Cauchy's integral formula is valid, and we have

$$\begin{aligned} |f^{(n)}(z)| &= \left| \frac{n!}{2\pi i} \int_{|z|=1} \frac{f(\zeta)\, d\zeta}{(\zeta - z)^{n+1}} \right| \leq \frac{n!}{2\pi} \int_{|z|=1} \frac{|d\zeta|}{(|\zeta| - |z|)^{n+1}} \\ &= \frac{n!}{2\pi} \int_{|z|=1} \frac{|d\zeta|}{(1-|z|)^{n+1}} = \frac{n!}{(1-|z|)^{n+1}} \end{aligned}$$

(c) Suppose that f and g are both analytic on the set S, and that their real parts are equal on the boundary of S. That is, if $|z| = 1$ then $\mathrm{Re}(f(z)) = \mathrm{Re}(g(z))$. What can you say about f and g in this situation?

Set $h = f - g$. Then h is an analytic function whose real part is zero on $|z| = 1$. Thus, by the maximum and minimum modulus principles for harmonic functions, the real part of h must be zero in S. From the Cauchy-Riemann equations we see that the gradient of the imaginary part of h must be zero. That is, the imaginary part of h is some constant c, which means that $h = ci$, or the functions f and g differ by a purely imaginary constant.

3. (40) Evaluate the following integrals.

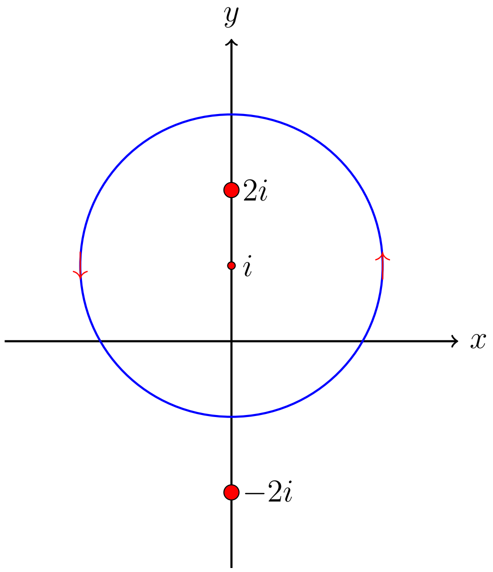
(a) $\displaystyle\int_\Gamma (z^2 + \sin z)dz$, where Γ is the path which goes from 0 to 1, and then to $1+i$ along straight lines.

The integrand is the derivative of the entire function $\dfrac{z^3}{3} - \cos z$. Thus, the integral does not depend on the path of integration, but only on the endpoints.

$$\int_0^{1+i} (z^2 + \sin z)dz = \left.\frac{z^3}{3} - \cos z\right|_0^{1+i} = \frac{(1+i)^3}{3} - \cos(1+i) + 1\,.$$

(b) $\displaystyle\int_{|z-i|=2} \frac{dz}{z^2+4}$

The path of integration and the singular points of the integrand (the large red dots) are shown below.



$$\int_{|z-i|=2} \frac{dz}{z^2+4} = \int_{|z-i|=2} \frac{\frac{1}{z+2i}}{z-2i}\, dz = 2\pi i \left. \frac{1}{z+2i} \right|_{z=21} = \frac{2\pi i}{4i} = \frac{\pi}{2}$$

(c) $\displaystyle\int_{|z|=1} \frac{e^{kz}}{z}\, dz$, where k is any constant.

The function e^{kz} is entire so Cauchy's integral formula can be used to evaluate this integral

$$\int_{|z|=1} \frac{e^{kz}}{z}\, dz = 2\pi i \left. e^{kz} \right|_{z=0} = 2\pi i\,.$$