1. (25) Show that the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges normally to e^z in the complex plane.

If F is any compact subset of $\mathbb{C}$, then there is an $R > 0$ such that $F \subset \{z : |z| \leq R\}$. We'll show that the series converges uniformly on sets of the form $\{z : |z| \leq R\}$, which then implies that the series converges normally in the complex plane. Let $|z| \leq R$, then we have

$$\begin{aligned} e^z &= \frac{1}{2\pi i} \int\limits_{|\zeta|=2R} \frac{e^\zeta}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int\limits_{|\zeta|=2R} \frac{e^\zeta}{\zeta} \frac{1}{1 - z/\zeta}\, d\zeta \\ &= \frac{1}{2\pi i} \int\limits_{|\zeta|=2R} \frac{e^\zeta}{\zeta} \left[\sum_{n=0}^{\infty} \left(\frac{z}{\zeta} \right)^n \right] d\zeta \end{aligned}$$

We know that the geometric series converges normally on the open unit disc. Since $|z/\zeta| \leq 1/2$ the series converges uniformly to the function $\frac{1}{1-z/\zeta}$ on the set $\{z : |z| \leq R\} \times \{\zeta : |\zeta| = 2R\}$. Moreover, since the series converges uniformly we may interchange the order of integration and summation to get the desired equality.

$$e^z = \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \int\limits_{|\zeta|=2R} \frac{e^\zeta}{\zeta^{n+1}} d\zeta \right] z^n = \sum_{n=0}^{\infty} \frac{\left.\frac{d^n(e^\zeta)}{d\zeta^n}\right|_{\zeta=0}}{n!} z^n = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

2. (20) Let $f(z) = \frac{1}{z^4}(z - \sin z)$.

   (a) Find the Laurent series expansion of f about $z = 0$. What type of singular point is 0? For which z does the Laurent series converge?

   We know that $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$ and the series converges normally in the complex plane. Thus

$$\begin{aligned} \frac{1}{z^4}(z - \sin z) &= \frac{z - (z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots)}{z^4} = \frac{\frac{z^3}{3!} - \frac{z^5}{5!} + \cdots}{z^4} \\ &= \frac{1}{3!z} - \frac{z}{5!} + \cdots. \end{aligned}$$

   The series converges for $|z| > 0$, and the origin is a pole of order 1.

(b) Let C_2 denote the positively oriented circle of radius 2 centered at the origin. Calculate $\int_{C_2} f(z)\, dz$.

The function f is the sum of $\frac{1}{3!z}$ and an analytic function. The integral of f over the circular path is then the sum of the integrals of $\frac{1}{3!z}$ and the analytic function over this path. The integral of the analytic part will be zero. So we'll have

$$\int\limits_{C_2} f(z)\, dz = \int\limits_{C_2} \frac{1}{3!z}\, dz = \frac{1}{6}\int\limits_{C_2} \frac{1}{z}\, dz = \frac{2\pi i}{6} = \frac{\pi i}{3}\,.$$

3. (25) Let $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n2^n}$

(a) Find the radius of convergence of this series.

$$\lim_{n\to\infty}\left|\frac{z^{n+1}/(n+1)2^{n+1}}{z^n/n2^n}\right| = \frac{|z|}{2}\lim_{n\to\infty}\frac{n}{n+1} = \frac{|z|}{2}\,.$$

This limit must be less than 1, which tells us that $|z| < 2$ and the radius of convergence is 2.

(b) $f(1) =$? Note: the answer $f(1) = \sum_{n=1}^{\infty} \frac{1}{n2^n}$ is not sufficient. Hint: find a closed form expression for f.

If we differentiate the given series we can find a closed form expression for the derivative of f.

$$f'(z) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{2^n} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{z}{2}\right)^{n-1} = \frac{1}{2-z}$$

Thus $f(x) = -\log(2-z) + c$ for some branch of the log function, and c is a constant. $f(0) = 0$ implies $c = \log 2$, and since $f(x)$ is positive for $0 < x < 2$, we have $f(x) = \ln 2 - \ln(2-x)$ for $0 < x < 2$. Thus,

$$f(1) = \sum_{n=1}^{\infty} \frac{1}{n2^n} = \ln 2 - \ln(2-1) = \ln 2\,.$$

4. (15) Assume that $f_n(z)$ are continuous functions of z in a domain D, and assume that the sequence of functions converge normally to f in D. Let Γ be a piecewise smooth curve in D. Show that

$$\lim_{n\to\infty} \int_\Gamma f_n(z)dz = \int_\Gamma f(z)dz$$

The curve Γ is a compact subset of D. Thus, the functions f_n converge uniformly to f on Γ, and since each f_n is continuous so is the limit function, which means that the integral of f over Γ exists. Let $\epsilon > 0$, let L denote the length of the curve Γ. Pick N so that if $n > N$ we have $|f_n(z) - f(z)| < \frac{\epsilon}{L}$ for all $z \in \Gamma$. Then for $n > N$ we have

$$\left| \int_\Gamma f_n(z)\,dz - \int_\Gamma f(z)\,dz \right| \le \int_\Gamma |f_n(z) - f(z)|\,|dz| \le \frac{\epsilon}{L} L = \epsilon\,.$$

5. (15) Suppose $f(z) = \displaystyle\sum_{k=-\infty}^{\infty} a_k z^k$, and the series converges for $0 < |z| < 1$.

Let $g(z) = f(z) - \dfrac{a_{-1}}{z}$. Show that $g(z)$ is the derivative of an analytic function defined on $0 < |z| < 1$.

The first thing to note is that the function $z^k, k \neq -1$, is the derivative of the function $\frac{z^{k+1}}{k+1}$, and this function is entire if $k \ge 0$ and analytic for $|z| > 0$ if $k \le -2$. Thus, we have

$$g(z) = f(z) - \frac{a_{-1}}{z} = \sum_{\substack{k=-\infty \\ k\neq -1}}^{\infty} a_k z^k = \sum_{\substack{k=-\infty \\ k\neq -1}}^{\infty} a_k \frac{d}{dz}\frac{z^{k+1}}{k+1} = \frac{d}{dz} \sum_{\substack{k=-\infty \\ k\neq -1}}^{\infty} a_k \frac{z^{k+1}}{k+1}\,.$$

Moreover, the ratio test shows that the series $\sum_{\substack{k=-\infty \\ k\neq -1}}^{\infty} a_k \frac{z^{k+1}}{k+1}$, converges for $0 < |z| < 1$. This follows from the fact that the series $\sum_{k=-\infty}^{-1} a_k z^k$ converges for $|z| > 0$, and the series $\sum_{k=0}^{\infty} a_k z^k$ converges for $|z| < 1$.